Advanced Quantum Information and Computing

Exercise Sheet 3

Exercise 1 (About von Neumann entropy)**.** *The von Neumann entropy of a quantum system, expressed as density operator ρ is (with the convention $0 \log 0 = 0$)*

$$S(\rho) \overset{def}{=} -\operatorname{tr}(\rho \log \rho)$$

1. *Why is $S(\rho)$ well defined? Give the expression of $S(\rho)$ according to the eigenvalues of ρ,*
2. *Give the entropy of the states $\rho_0 \overset{def}{=} |0\rangle\langle 0|$ and $\rho_1 \overset{def}{=} \frac{|0\rangle\langle 0|+|1\rangle\langle 1|}{2}$,*
3. *Prove that the von Neumann entropy of pure states is 0,*
4. *Give the entropy of the probabilistic mixture of $|+\rangle$ with prob. $\frac{1}{2}$ and $|-\rangle$ with prob. $\frac{1}{2}$,*
5. *Prove that the von Neumann entropy of ρ is zero if and only ρ is a pure state. What happens if ρ is a mixed quantum state?*
6. *Suppose that p_i are probabilities, and the states ρ_i have support on orthogonal subspaces. Show that (where H is the classical entropy),*

$$S\left(\sum_i p_i \rho_i\right) = H\left((p_i)_i\right) + \sum_i p_i S(\rho_i)$$

7. **Joint entropy***: suppose p_i are probabilities, $|i\rangle$ are orthogonal states for a system A, and ρ_i is any set of density operators for another system, B. Show that,*

$$S\left(\sum_i p_i |i\rangle\langle i| \otimes \rho_i\right) = H\left((p_i)_i\right) + \sum_i p_i S(\rho_i)$$

Exercise 2 (Klein's inequality)**.** *An important quantity in information theory is the relative entropy, also known as the Kullback-Leibler divergence. Given two distributions $(p_i)_i$ and $(q_i)_i$*

$$D_{\mathrm{KL}}(p_i||q_i) \overset{def}{=} \sum_i p_i \ \log_2 \frac{p_i}{q_i} \in \mathbb{R} \cup \{+\infty\}$$

This quantity is often useful, not in itself, but because other entropy quantities can be regarded as a special case of $D_{\mathrm{KL}}(\cdot||\cdot)$*. The most useful fact is the following inequality (known as Gibb's inequality),*

$$D_{\mathrm{KL}}(p_i||q_i) \geq 0 \quad \textit{and} \quad D_{\mathrm{KL}}(p_i||\sigma_i) = 0 \iff p_i = q_i \textit{ for all } i$$

In a similar vein it is also extremely useful to define a quantum version of the relative entropy. Suppose ρ *and* σ *are density operators. The relative entropy of* ρ *to* σ *is defined by*

$$S(\rho||\sigma) = \mathrm{tr}(\rho \log_2 \rho) - \mathrm{tr}(\rho \log_2 \sigma)$$

As with the classical relative entropy, the quantum relative entropy can sometimes be infinite. In particular, the relative entropy is defined to be $+\infty$ *if the kernel of* σ *(the vector space spanned by the eigenvectors of* σ *with eigenvalue 0) has non-trivial intersection with the support of* ρ *(the vector space spanned by the eigenvectors of* ρ *with non-zero eigenvalue), and is finite otherwise.*

1. *Show that* $S(\rho||\sigma) \geq 0$ *with equality if and only if* $\rho = \sigma$*. This inequality is known as Klein's inequality. It is useful to prove many statements involving von Neumann entropy.*

2. *Deduce that the von Neumann entropy is equal to* $\log_2 d$ *(where* d *is the dimension of the underlying Hilbert space) if and only if the system is in the completely mixed state* $\mathbf{I}/d$

Exercise 3 (Entropy of a tensor product)**.** *Use the joint entropy property of von Neumann entropy to show that*

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$$

Prove this result directly from the definition of the von Neumann entropy.

Exercise 4 (Projective measurements increase entropy)**.** *Suppose* $\mathbf{P}_i$ *is a complete set of orthogonal projectors and* ρ *is a density operator. Then the entropy of the state* $\rho' = \mathbf{P}_i \rho P i$ *of the system after the measurement is at least as great as the original entropy,*

$$S(\rho') \geq S(\rho)$$

with equality if and only if $\rho = \rho'$*.*

Exercise 5. *Compute the von Neumann entropy of*

$$\rho_0 \stackrel{def}{=} (1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1| \quad \textit{and} \quad \rho_1 \stackrel{def}{=} (1-p)\,|0\rangle\langle 0| + p\,|+\rangle\langle +|$$

What do you deduce according to the lecture? (draw a picture with some ***`script`****)*