Information Theory

Exercise Sheet 5

Exercise 1 (About the binomial)**.**

1. *Show with the method of types that,*

$$\frac{1}{(n+1)^2} 2^{nh(k/n)} \leq \binom{n}{k} \leq 2^{nh(k/n)}$$

2. *Our main now is to sharpen the lower-bound by showing that*

$$\frac{1}{n+1} 2^{nh(k/n)} \leq \binom{n}{k} \leq 2^{nh(k/n)}$$

Hint: *first show that the most likely event of a binomial distribution of parameters n, p is np*

Exercise 2 (Relation between $D_{\mathrm{KL}}(P||Q)$ and chi-square)**.** *Show that the χ^2 statistic*

$$\chi^2 = \sum_x \frac{(P(x) - Q(x))^2}{Q(x)}$$

is twice the first term in the Taylor series expansion of $D_{\mathrm{KL}}(P||Q)$ about Q.

Hint: *Write $P/Q = 1 + (P - Q)/Q$ and expand the* $\log_2$

Exercise 3 (Error exponent for universal codes)**.**

1. *Show that the universal source code of lecture with rate R achieves a probability of error $P_e^{(n)} \overset{(poly)}{=} 2^{-nD_{\mathrm{KL}}(P^\star||Q)}$, where Q is the true distribution and $P^\star$ achieves* $\min D_{\mathrm{KL}}(P||Q)$ *over all the P's such that $H(P) \geq R$*

2. *Find $P^\star$ in terms of Q and R*

3. *Now let $\mathbf{X} \in \{0, 1\}$. Find the region of source probabilities $Q(x)$ for which rate R is sufficient for the universal source code to achieve $P_e^{(n)} \longrightarrow 0$.*

Exercise 4. *Consider a joint distribution $Q(x,y)$ with marginals $Q(x)$ and $Q(y)$. Let E be the set of types that look jointly typical with respect to Q:*

$$
\begin{aligned}
E \overset{def}{=} \Big\{ P(x,y) \in \mathcal{P} : & \left| -\sum_{x,y} P(x,y) \log_2 Q(x) - H(\mathbf{X}) \right| = 0, \\
& \left| -\sum_{x,y} P(x,y) \log_2 Q(y) - H(\mathbf{Y}) \right| = 0, \\
& \left| -\sum_{x,y} P(x,y) \log_2 Q(x,y) - H(\mathbf{X},\mathbf{Y}) \right| = 0 \Big\}
\end{aligned}
$$

1. *Let $Q_0(x,y)$ be another distribution on $\mathcal{X} \times \mathcal{Y}$. Argue that the distribution $P^\star$ in E that is the closest to Q_0 is of the form*

$$P^\star(x,y) = Q_0(x,y) 2^{\lambda_0 + \lambda_1 \log_2 Q(x) + \lambda_2 \log_2 Q(y) + \lambda_3 \log_2 Q(x,y)} \tag{1}$$

   where $\lambda_0, \lambda_1, \lambda_2$ and λ_3 are chosen to satisfy the constraints. Argue that this distribution is unique.

2. *Now let $Q_0(x,y) = Q(x)Q(y)$. Verify that $Q(x,y)$ is of the form (1) and that it belongs to E. Deduce that $P^\star = Q$.*

Exercise 5 (Counting)**.** *Let $\mathcal{X} = \{1, 2, \ldots, m\}$. Show that the number of sequences $\mathbf{x} \in \mathcal{X}^n$ satisfying $1/n \sum_{i=1}^n g(x_i) \geq \alpha$ is approximately equal to $2^{nH^\star}$, to first order in the exponent, for n sufficiently large, where*

$$H^\star = \max_{P : \sum_{i=1}^m P(i) g(i) \geq \alpha} H(P)$$

Exercise 6 (Running difference)**.** *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d. according to Q_1, and $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be i.i.d. according to Q_2. Let $\mathbf{X}^n$ and $\mathbf{Y}^n$ be independent. Find an expression for*

$$\mathbb{P}\left(\sum_{i=1}^n \mathbf{X}_i - \mathbf{Y}_i \geq nt \right)$$

good to first order in the exponent.

Exercise 7 (Type constraint)**.**

1. *Find constraints on the type* $P^{\text{emp}}_{\mathbf{X}^n}$ *such that the sample variance verifies*

$$\overline{\mathbf{X}^2_n} - \left(\overline{\mathbf{X}_n}\right)^2 \leq \alpha$$

where

$$\overline{\mathbf{X}^2_n} \overset{def}{=} \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^2 \quad \text{and} \quad \overline{\mathbf{X}_n} \overset{def}{=} \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i$$

2. *Find the exponent in the probability* $Q^n\left(\overline{\mathbf{X}^2_n} - \left(\overline{\mathbf{X}_n}\right)^2 \leq \alpha\right)$.