QUANTUM OBLIVIOUS LWE SAMPLING AND INSECURITY OF STANDARD MODEL LATTICE-BASED SNARKS

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A mod-q linear system "with errors" to solve

LWE(m, n, q, σ): Learning With Errors

- **Input**: $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e})$ where,

$$\mathbf{A} \xleftarrow{unif} (\mathbb{Z}/q\mathbb{Z})^{m \times n}, \quad \mathbf{s} \xleftarrow{unif} (\mathbb{Z}/q\mathbb{Z})^{n} \quad \textbf{and} \quad \mathbf{e} \longleftarrow \mathcal{D} \quad \text{s.t. } |e_i| \approx \sigma$$

- **Output**: $\mathbf{s}$

$\longrightarrow$ Distribution $\mathcal{D}$ ensures small coefficients for $\mathbf{e}$

*Parameters m, n, q, σ are chosen to ensure **unicity of the solution $\mathbf{s}$***

$(\mathbf{s}, \mathbf{e}) \mapsto \mathbf{A}\mathbf{s} + \mathbf{e} \in (\mathbb{Z}/q\mathbb{Z})^m$ is **sparse** in its range

LWE *is conjectured as being hard on average*

even in the quantum computational model

LWE Source of Hardness:

- enjoys self-reducibility (as hard as its worst-case variant)
- no easier than computing short vector in a lattice (Regev quantum reduction)

LWE hardness ensures the security of some:

- Encryption schemes,
- Fully Homomorphic Encryption schemes

$\longrightarrow$ LWE *is a very versatile problem to design cryptographic primitives*

Variants of the LWE*-hardness have been introduced*

to design some advanced cryptographic primitives

Assumption: Efficient Oblivious LWE-Sampling is Impossible

Every algorithm generating LWE samples $\left(\mathbf{A}, \mathbf{b} \stackrel{\text{def}}{=} \mathbf{A}\mathbf{s} + \mathbf{e}\right)$

knows the underlying secret $\mathbf{s}$

- Assumption used [GMNO18, NYI+20, ISW21, SSEK22, CKKK23, GNSV23] to ensure security of some lattice SNARK (Succinct Non-interactive Arguments of Knowledge)

OUR RESULT

An Oblivious Sampler for LWE:

A **quantum algorithm** generating LWE samples $\left(\mathbf{A}, \mathbf{b} \stackrel{\text{def}}{=} \mathbf{A}\mathbf{s} + \mathbf{e}\right)$ without knowing $\mathbf{s}$

$\longrightarrow$ The only way to extract $\mathbf{s}$ from the sampler is to solve the LWE-problem $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e})$

- Our quantum oblivious sampler takes advantage of:
  - **complex phases** in quantum computation
  - an optimal **unambiguous quantum measurement**
- First application: it invalidates security proofs of some lattice-based SNARK but does not break them

ROADMAP

1. A Fundamental Quantum State to Build
2. Quantum Unambiguous Measurement
3. Quantum Oblivious LWE Sampler

A FUNDAMENTAL QUANTUM STATE

$$\mathbf{e} \leftarrow \mathrm{Gauss}(\sigma)^{\otimes m}\text{: the } e_i \text{ are i.i.d and } \mathbb{P}(e_i = x) = \mathrm{Gauss}(\sigma)(x) = \frac{e^{-\pi x^2/\sigma^2}}{\sigma}$$

$$\left(\text{ up to normalization mod } q\right)$$

LWE(m, n, q, σ): Learning With Errors

- **Input**: $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ where,

$$\mathbf{A} \xleftarrow{unif} (\mathbb{Z}/q\mathbb{Z})^{m \times n}, \quad \mathbf{s} \xleftarrow{unif} (\mathbb{Z}/q\mathbb{Z})^{n} \quad \text{and} \quad \mathbf{e} \longleftarrow \mathrm{Gauss}(\sigma)^{\otimes m}$$

- **Output**: $\mathbf{s}$

Key Idea:

To achieve oblivious LWE sampling,

$$(i) \text{ build } \sum_{\mathbf{s},\mathbf{e}} \left(\prod_i \sqrt{\mathrm{Gauss}(\sigma)(e_i)} \right) |\mathbf{As} + \mathbf{e}\rangle \quad \text{and} \quad (ii) \text{ measure}$$

Is it hard to build $\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i \sqrt{\mathrm{Gauss}(\sigma)(e_i)} \right) |\mathbf{As}+\mathbf{e}\rangle$?

Quantum Regev Reduction in a Nutshell:

(i) $\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i \sqrt{\mathrm{Gauss}(\sigma)(e_i)} \right) |\mathbf{As}+\mathbf{e}\rangle \xrightarrow{\mathrm{QFT}} \sum_{\mathbf{x}:\ \mathbf{A}^\top \mathbf{x}=0} \left(\prod_i \sqrt{\mathrm{Gauss}(4/\sigma)(x_i)} \right) |\mathbf{x}\rangle$

(ii) Then measuring gives a short $\mathbf{x}_0$ in the lattice $\{\mathbf{x} \in \mathbb{Z}^m : \mathbf{A}^\top \mathbf{x} = \mathbf{0} \bmod q\}$

$\longrightarrow$ Building $\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i \sqrt{\mathrm{Gauss}(\sigma)(e_i)} \right) |\mathbf{As}+\mathbf{e}\rangle$ implies the ability to compute a short vector in a lattice **which is a hard problem. . .**

Fundamental Remark: Adding Phases

Considering $\sum_{\mathbf{s},\mathbf{e}} \lambda_{\mathbf{s},\mathbf{e}} \left(\prod_i \sqrt{\mathrm{Gauss}(\sigma)(e_i)} \right) |\mathbf{A}\mathbf{s} + \mathbf{e}\rangle$ where $\lambda_{\mathbf{s},\mathbf{e}} \in \mathbb{C}$ and $|\lambda_{\mathbf{s},\mathbf{e}}| = 1$

- Measuring with phases still gives a quantum oblivious LWE sampler
- Measuring after applying QFT does not necessarily give a short lattice vector

$$\mathrm{QFT}\left(\sum_{\mathbf{s},\mathbf{e}} \lambda_{\mathbf{s},\mathbf{e}} \left(\prod_i \sqrt{\mathrm{Gauss}(\sigma)(e_i)} \right) |\mathbf{A}\mathbf{s} + \mathbf{e}\rangle \right) \neq \sum_{\mathbf{x}:\mathbf{A}^\top \mathbf{x} = 0} \prod_i \sqrt{\mathrm{Gauss}(4/\sigma)(x_i)}\, |\mathbf{x}\rangle$$

QUANTUM UNAMBIGUOUS MEASUREMENT

Naive Approach to Build $\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |\mathbf{A}\mathbf{s}+\mathbf{e}\rangle$:

- Build,

$$\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |\mathbf{s}\rangle\, |\mathbf{e}\rangle \quad \left(f \text{ is efficiently computable} \right)$$

- Multiplication by $\mathbf{A}$ and add to the second register,

$$\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |\mathbf{s}\rangle\, |\mathbf{A}\mathbf{s}+\mathbf{e}\rangle$$

- Disentangle **by applying an LWE-solver**, *i.e.*, $\mathcal{A}\left(\mathbf{A}\mathbf{s}+\mathbf{e}\right) \mapsto \mathbf{s}$,

$$\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |\mathbf{s}-\mathcal{A}\left(\mathbf{A}\mathbf{s}+\mathbf{e}\right)\rangle\, |\mathbf{A}\mathbf{s}+\mathbf{e}\rangle = \sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |0\rangle\, |\mathbf{A}\mathbf{s}+\mathbf{e}\rangle$$

$\longrightarrow$ **Not efficient**: it relies on an LWE-solver

[CLZ22][(1)] proposed a new approach to build

$$\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |\mathbf{As} + \mathbf{e}\rangle$$

$\longrightarrow$ Unambiguous measurement to disentangle $\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |\mathbf{s}\rangle \, |\mathbf{As} + \mathbf{e}\rangle$

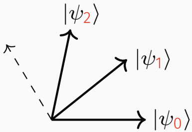
(1) Yilei Chen, Qipeng Liu, and Mark Zhandry. *Quantum algorithms for variants of average-case lattice problems via filtering*. In EUROCRYPT, 2022.

PREVIOUS WORK: [CLZ22] AND UNAMBIGUOUS MEASUREMENT

Given $\mathbf{A} = (\mathbf{a}_1 | \cdots | \mathbf{a}_m)^\top$ and denoting $\mathbf{x} \cdot \mathbf{y} \stackrel{\text{def}}{=} \sum_i x_i y_i \in \mathbb{Z}/q\mathbb{Z}$

$$\sum_{\mathbf{s},\mathbf{e}} \left(\prod_i f(e_i) \right) |\mathbf{s}\rangle\, |\mathbf{A}\mathbf{s} + \mathbf{e}\rangle = \sum_{\mathbf{s}} |\mathbf{s}\rangle \bigotimes_i \left(\underbrace{\sum_{e_i} f(e_i)\, |\mathbf{a}_i \cdot \mathbf{s} + e_i\rangle}_{\stackrel{\text{def}}{=} \left|\psi_{\mathbf{a}_i \cdot \mathbf{s}}\right\rangle} \right)$$

$$\forall j \in \mathbb{Z}/q\mathbb{Z}, \quad |\psi_j\rangle \stackrel{\text{def}}{=} \sum_{e \in \mathbb{Z}/q\mathbb{Z}} f(e)\, |j + e\rangle$$



Key-Idea: Quantum Unambiguous Measure

$$\left|\psi_{\mathbf{a}_i \cdot \mathbf{s}}\right\rangle \xrightarrow[\text{measure}]{\text{unambiguous}} \begin{cases} \mathbf{a}_i \cdot \mathbf{s} & \text{with probability } p \\ \perp & \text{with probability } 1 - p \end{cases}$$

Using a quantum unambiguous measure reduces to solve a linear system **with erasure**

QUANTUM OBLIVIOUS LWE SAMPLER

AN OPTIMAL QUANTUM UNAMBIGUOUS MEASUREMENT

$$\sum_{\mathbf{s}} |\mathbf{s}\rangle \left(\sum_{e_1} f(e_1) \, |\mathbf{a}_1 \cdot \mathbf{s} + e_1\rangle \right) \otimes \cdots \otimes \left(\sum_{e_m} f(e_m) \, |\mathbf{a}_m \cdot \mathbf{s} + e_m\rangle \right)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\perp \qquad\qquad\qquad\qquad \mathbf{a}_m \cdot \mathbf{s}$$

We succeed to recover $\mathbf{a}_i \cdot \mathbf{s}$ with probability p

We are successful on $\approx pm$ coordinates: it is necessary that $pm \geq n$ to recover $\mathbf{s} \in (\mathbb{Z}/q\mathbb{Z})^n$

- In [CLZ22]: $p^{\mathrm{CLZ}} = \frac{\min_x |\widehat{f}(x)|^2}{q}$
- Optimal unambiguous measurement [CB98][2]: $p^{\mathrm{CB}} = q \cdot \min_x |\widehat{f}(x)|^2$

[2] Anthony Chefles and Stephen M. Barnett. *Optimum unambiguous discrimination between linearly independent symmetric states.* Phys. Lett. A, 1998.

ADD PHASES

Our quantum algorithm **uses m registers** with $m = \frac{n}{p^{\mathrm{CB}}} = \frac{n}{q \cdot \min_x |\widehat{f}(x)|^2}$

Issue:

$$\text{If } f = \sqrt{\mathrm{Gauss}(q, \sigma)}, \quad \text{then} \quad \widehat{f} = \mathrm{Gauss}(2/\sigma)$$

$$m = \frac{n}{q \cdot \min_x |\widehat{f}(x)|^2} = \mathrm{e}^{\Omega(n)}$$

Key-Idea: Use Phases

$$f(x) = \begin{cases} \sqrt{\mathrm{Gauss}(\sigma)(x)} & \text{if } x > 0 \\ (-1) \cdot \sqrt{\mathrm{Gauss}(\sigma)(x)} & \text{otherwise} \end{cases}$$

Then,

$$m = \frac{n}{q \cdot \min_x |\widehat{f}(x)|^2} \leq \frac{n}{\mathrm{Gauss}(\sigma)(0)} \approx n \cdot \sigma$$

$\Big($with measurement from [CLZ22]: $m = q^2 \cdot n \cdot \sigma = \mathrm{e}^{\Omega(n)}$ when $q = \mathrm{e}^{\Omega(n)}\Big)$

SUMMARY

Theorem:

Parameters m, n, q, σ are functions of λ and they satisfy,

$$q \text{ prime}, \quad m, \log q \leq \mathrm{poly}(\lambda), \quad m \geq n\sigma \cdot \omega(\log \lambda) \quad \text{and} \quad 2 \leq \sigma \leq \frac{q}{\sqrt{8m \ln q}} .$$

Then, there exists a $\mathrm{poly}(\lambda)$-time quantum oblivious $\mathrm{LWE}(m, n, q, \sigma)$ instance sampler, under the assumption that $\mathrm{LWE}(m, n, q, \sigma)$ is hard.

$\longrightarrow$ To reach other parametrizations (σ larger, q not prime, etc. . .)

we use reductions (modulus switching, noise flooding) conserving obliviousness

CONCLUSION

Our result: a quantum algorithm which **obliviously** samples (given $\mathbf{A}$),

$$\mathbf{A}\mathbf{s}+\mathbf{e} \quad \text{with} \quad \mathbf{s} \xleftarrow{unif} (\mathbb{Z}/q\mathbb{Z})^n \quad \text{and} \quad \mathbf{e} \leftarrow \mathrm{Gauss}(\sigma)^{\otimes m}$$

What we did not discuss:

- Definition of classical and quantum oblivious sampling
- How to efficiently run the unambiguous measurement from [CB98]
- Why does it invalidate the security proofs of some SNARKs

Future Work:

Is this oblivious LWE-sampler can be used to design advanced quantum protocols?

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