INF587 Exercise sheet 3

Exercise 1 (Bloch sphere)**.**

1. *Why does any qubit can be written as*

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

The number θ and φ define a point on the unit three-dimensional sphere as:

$$(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

It is shown in Figure 1: it is the Bloch sphere representation.

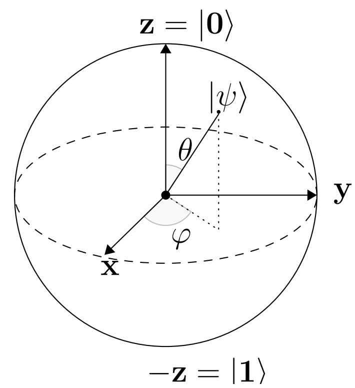


Figure 1: Bloch sphere.

2. *How are the Bloch representations of orthogonal qubits?*

This description has an important generalization to mixed states, the Bloch ball representation*, as follows.*

3. *Show that an arbitrary density matrix for a mixed state qubit may be written as*

$$\rho = \frac{\mathbf{I}_2 + \mathbf{v}\cdot\sigma}{2}$$

where $\mathbf{v} \in \mathbb{R}^3$ *has Euclidean norm* ≤ 1*. Here* $\mathbf{v}\cdot\sigma \overset{def}{=} \sum_{i=1}^{3} v_i\sigma_i = v_1\mathbf{X} + v_2\mathbf{Y} + v_3\mathbf{Z}$ *where recall that*

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The vector $\mathbf{v}$ *is known as the Bloch vector for the state* ρ *and it gives the representation of the state as a point in the unit ball.*

Hint: *treat first the case where* ρ *is a pure state.*

4. *Show that for pure states the description of the Bloch vector we have given coincides with that of question* 1.

5. *What is the Bloch vector representation for the state* $\rho \stackrel{def}{=} \mathbf{I}_2/2$*?*

6. *Show that a state* ρ *is pure if and only if* $\|\mathbf{v}\|_2 = 1$.

Exercise 2 (von Neumann entropy)**.** *The von Neumann entropy of a quantum system, expressed as density operator* ρ *is (with the convention* $0 \log 0 = 0$*)*

$$S(\rho) \stackrel{def}{=} -\operatorname{tr}(\rho \log \rho)$$

1. *Why is* $S(\rho)$ *well defined? Give the expression of* $S(\rho)$ *according to the eigenvalues of* ρ,

2. *Give the entropy of the states* $\rho_0 \stackrel{def}{=} |0\rangle\langle 0|$ *and* $\rho_1 \stackrel{def}{=} \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$,

3. *Prove that the von Neumann entropy of pure states is* 0,

4. *Give the entropy of the probabilistic mixture of* $|+\rangle$ *with prob.* $\frac{1}{2}$ *and* $|-\rangle$ *with prob.* $\frac{1}{2}$,

5. *Prove that the von Neumann entropy of* ρ *is zero if and only* ρ *is a pure state. What happens if* ρ *is a mixed quantum state?*

Exercise 3. *Suppose a composite of systems* A *and* B *is in the state* $|a\rangle|b\rangle$, *where* $|a\rangle$ *is a pure state of* A, *and* $|b\rangle$ *is a pure state of* B. *Show that the reduced density operator of system* A *alone is a pure state.*

Exercise 4. *For each state* $|\psi_{AB}\rangle$, *give the reduced density matrices*

$$\rho_A = \operatorname{tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) \quad \textit{and} \quad \rho_B = \operatorname{tr}_A(|\psi_{AB}\rangle\langle\psi_{AB}|).$$

You can write your answers in Dirac's "ket,bra" notation or in matrix form. Compute also $S(\rho_A)$ *in each case (see Exercise 2). You can use* $\log_2(3) \approx 1.585$.

1. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|-\rangle + |1\rangle|+\rangle\right)$.

2. $|\psi_{AB}\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$.

3. $|\psi_{AB}\rangle = \sqrt{\frac{3}{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle - \sqrt{\frac{1}{8}}|10\rangle + \sqrt{\frac{1}{8}}|11\rangle$.

Exercise 5 (Schmidt decomposition)**.** *Find the Schmidt decomposition and give the Schmidt number of the following two qubits:*

$$|\psi_1\rangle \stackrel{def}{=} \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle \stackrel{def}{=} \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$|\psi_3\rangle \stackrel{def}{=} \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2}$$

Exercise 6. *Prove that a state $|\psi\rangle$ of a composite system $A \otimes B$ is a product state if and only if it has Schmidt number* 1*. Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states where $\rho = |\psi\rangle\langle\psi|$ and $\rho_A = \mathrm{tr}_B(\rho)$ and $\rho_B = \mathrm{tr}_A(\rho)$.*

Deduce the theorem of the lecture: a pure state $|\psi\rangle \in A \otimes B$ is entangled if and only if its Schmidt's number is > 1 if and only if ρ_A and ρ_B (defined as above) are mixed states.

Exercise 7 (Bell states)**.** *The four Bell states are defined as*

$$|\beta_{00}\rangle \stackrel{def}{=} \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{01}\rangle \stackrel{def}{=} \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle \stackrel{def}{=} \frac{|00\rangle - |11\rangle}{\sqrt{2}} \quad \textit{and} \quad |\beta_{11}\rangle \stackrel{def}{=} \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

For each of the four Bell states, find the reduced density operator for each qubit.

Exercise 8. *Suppose $\{p_i, |\psi_i\rangle\}$ is an ensemble of states generating a density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ for a quantum system A. Let R be a quantum system with orthonormal basis $(|i\rangle)$.*

1. *Show that $\sum_i \sqrt{p_i}\,|\psi_i\rangle|i\rangle$ is purification of ρ.*

2. *Suppose we measure R in the basis $(|i\rangle)$, obtaining the outcome i. With what probability do we obtain the result i, and what is the corresponding state of system A?*

3. *Let $|AR\rangle$ be any purification of ρ to the system $A \otimes R$. Show that there exists an orthonormal basis $(|i\rangle)$ in which R can be measured such that the corresponding post-measurement state for system A is $|\psi_i\rangle$ with probability p_i.*

Exercise 9 (Bell inequality)**.**

Imagine that both Alice and Bob have a particle. Alice can perform two measurements on her particle, for instance Alice can measure its speed or its position. Alice doesn't know in advance which measurement she will choose to perform. Rather, when she receives the particle she decides randomly which measurement to perform. The situation is the same for Bob, with two other possible measurements.

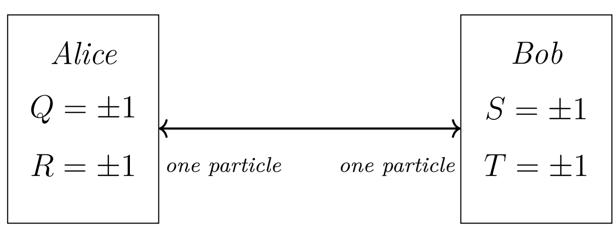
For the sake of simplicity, we suppose that measurements of Alice (resp. *Bob*) P_Q *and* P_R (resp. P_S *and* P_T) *have one of two outcomes* $Q, R \in \{-1, 1\}$ (resp. $S, T \in \{-1, 1\}$).

The timing of the experiment is so that Alice and Bob do their measurements at the same time. *Therefore, Alice's measurement cannot disturb Bob's measurement (and vice versa), since any physical influence cannot propagate faster than light.*

Alice and Bob perform many time this experiment and then meat together to use their common data to estimate the value of

$$QS + RS + RT - QT$$

Furthermore, we suppose that the two particles are prepared each time in the same way.



1. *Show that*

$$QS + RS + RT - QT = \pm 2.$$

Hint: *you may use* $QS + RS + RT - QT = (Q + R)S + (R - Q)T$.

2. Deduce that

$$\mathbb{E}\left(QS\right) + \mathbb{E}\left(RS\right) + \mathbb{E}\left(RT\right) - \mathbb{E}\left(QT\right) \leq 2.$$

The quantum experiment. Let $|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$*. Alice has the first qubit while Bob has the second one. Define:*

$$Q: \quad \text{meas. according to } \mathbf{Z}, \quad R: \quad \text{meas. according to } \mathbf{X}$$

$$S: \quad \text{meas. according to } \frac{-\mathbf{X}-\mathbf{Z}}{\sqrt{2}} \quad \text{and} \quad T: \quad \text{meas. according to } \frac{\mathbf{Z}-\mathbf{X}}{\sqrt{2}}$$

3. Show that

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2}.$$

4. Is it surprising?