INF587 Exercise sheet 2

Exercise 1. *Show that a normal operator/matrix is*

1. *Hermitian if and only if it has real eigenvalues,*
2. *Positive if and only if it has positive eigenvalues.*

Exercise 2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{P} \in \{$*Normal, Unitary, Hermitian, Projector, Positive*$\}$. *Show that* $\mathbf{A} \otimes \mathbf{B}$ *is* $\mathcal{P}$.

Exercise 3 (Exponential of Pauli matrices)**.** *Compute*

$$\exp(\theta \mathbf{X})$$

Let $\mathbf{v} \in \mathbb{R}^3$ *with Euclidean norm* 1 *and* $\theta \in \mathbb{R}$. *Show that*

$$\exp(i\theta \mathbf{v} \cdot \sigma) = \cos(\theta)\mathbf{I}_2 + i\sin(\theta)\mathbf{v} \cdot \sigma$$

where $\mathbf{v} \cdot \sigma \overset{def}{=} \sum_{i=1}^{3} v_i \sigma_i = v_1 \mathbf{X} + v_2 \mathbf{Y} + v_3 \mathbf{Z}$.

Hint: *compute* $(\mathbf{v} \cdot \sigma)^2$

Exercise 4 (Some projective measurements for qubits)**.**

1. *Show that* $\mathbf{X}, \mathbf{Y}$ *and* $\mathbf{Z}$ *are Hermitian and give their spectral decomposition in an orthonormal basis (eigenvalues with associated unit eigenvectors)*
2. *Suppose that we have a qubit in the state* $|0\rangle$, *and we measure the observable* $\mathbf{X}$. *What is the average value of* $\mathbf{X}$*? What is the standard deviation for* $\mathbf{X}$*?*
3. *Show that the average value of the observable* $\mathbf{X} \otimes \mathbf{Z}$ *for a two qubits system measured in the state* $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ *is zero.*
4. *Show that* $\mathbf{v} \cdot \sigma$ *(see Exercise 3) has eigenvalues* ± 1 *and that the projectors onto the corresponding eigenspaces are given by* $\mathbf{P}_{\pm 1} = \frac{(\mathbf{I}_2 \pm \mathbf{v} \cdot \sigma)}{2}$.

5. *Calculate the probability of obtaining the result* $+1$ *for a measurement of* $\mathbf{v} \cdot \sigma$ *given that the state prior ot measurement is* $|0\rangle$*. What is the state of the system after the measurement if* $+1$ *is obtained?*

Exercise 5 (About the POVM formalism)**.**

1. *Prove that no quantum measurement are capable of distinguishing non-orthogonal states.*

2*. *Give a* POVM $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ *that never makes error to distinguish the following quantum states:*

$$|\psi_1\rangle = |0\rangle \quad \textit{and} \quad |\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

Exercise 6 (projective measurements versus quantum measurements)**.** *Our aim in this exercise is to show that projective measurements together with unitary dynamics are sufficient to implement a general measurement. The rough idea is "to increase the dimension".*

Let $(\mathbf{M}_m)_{m \in \mathcal{M}}$ *be a quantum measurement that we want to perform on a state space* Q*. Notice that possible outcomes form a (finite) set* $\mathcal{M}$*.*

Let M *be an* ancilla system *with dimension* $\sharp\mathcal{M}$*. Let* $(|m\rangle)_{m \in \mathcal{M}}$ *be an orthonormal basis of* M*.*

1. *Let* $\mathbf{U}$ *be the following operator on* $Q \otimes M$ *(not linear as not defined over the whole space):*

$$\mathbf{U} : |\psi\rangle |0\rangle \in Q \otimes M \mapsto \sum_m \mathbf{M}_m |\psi\rangle |m\rangle$$

Show that:

$$\langle\varphi| \langle 0| \mathbf{U}^\dagger \mathbf{U} |\psi\rangle |0\rangle = \langle\varphi|\psi\rangle$$

2. *Show that* $\mathbf{U}$ *can be extended as a unitary operator on the space* $Q \otimes M$*.*

3. *Let* $\mathbf{P}_m \stackrel{\text{def}}{=} \mathbf{I}_Q \otimes |m\rangle\langle m|$*. Show that* $(\mathbf{P}_m)_{m \in \mathcal{M}}$ *is a projective measurement. In particular, given* $\mathbf{U} |\psi\rangle |0\rangle$*, what is the probability to outcome* m*? What becomes the* $\mathbf{U} |\psi\rangle |0\rangle$ *after measuring* m*?*

4. *Conclude.*

Exercise 7 (On Pauli matrices)**.**

1. *Let* $\mathbf{M} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$. *Show that it exists* $\alpha, \beta \in \mathbb{C}$ *such that* $\mathbf{M} = \alpha\mathbf{X} + \beta\mathbf{Y}$.

2. *Let* $\mathbf{M}$ *be any* 2×2 *complex matrix. Show that it exists* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *such that* $\mathbf{M} = \alpha\mathbf{I}_2 + \beta\mathbf{X} + \gamma\mathbf{Y} + \delta\mathbf{Z}$.

3. *Compute* $\mathbf{XZ}, \mathbf{XY}$ *and* $\mathbf{YZ}$. *Let* $\mathbf{P}_1, \mathbf{P}_2 \in \{\mathbf{I}_2, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. *Show that* $\mathrm{tr}(\mathbf{P}_1\mathbf{P}_2) = 0$ *if* $\mathbf{P}_1 \neq \mathbf{P}_2$ *and* $\mathrm{tr}(\mathbf{P}_1\mathbf{P}_2) = 2$ *if* $\mathbf{P}_1 = \mathbf{P}_2$.

4. *Let* $\mathbf{U}$ *be any unitary matrix on* 1 *qubit. We can hence write* $\mathbf{U} = \alpha\mathbf{I} + \beta\mathbf{X} + \gamma\mathbf{Y} + \delta\mathbf{Z}$. *Show that*
$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.$$

Exercise 8 (Heisenberg uncertainty principle)**.** *Given two Hermitian operators* $\mathbf{A}, \mathbf{B}$ *we define*

$$[\mathbf{A}, \mathbf{B}] \stackrel{def}{=} \mathbf{AB} - \mathbf{BA} \text{ (commutator)} \quad \text{and} \quad \{\mathbf{A}, \mathbf{B}\} \stackrel{def}{=} \mathbf{AB} + \mathbf{BA} \text{ (anti-commutator)}$$

1. *Show that*
$$| \langle\psi| [\mathbf{A}, \mathbf{B}] |\psi\rangle |^2 + | \langle\psi| \{\mathbf{A}, \mathbf{B}\} |\psi\rangle |^2 = 4| \langle\psi| \mathbf{AB} |\psi\rangle |^2$$
Deduce that
$$| \langle\psi| [\mathbf{A}, \mathbf{B}] |\psi\rangle |^2 \leq 4 \langle\psi| \mathbf{A}^2 |\psi\rangle \langle\psi| \mathbf{B}^2 |\psi\rangle$$

2. *Show that for two measurables* $\mathbf{C}$ *and* $\mathbf{D}$ *we have (Heisenberg uncertainty principle)*
$$\Delta(\mathbf{C})\,\Delta(\mathbf{D}) \geq \frac{|\langle\psi| [\mathbf{C}, \mathbf{D}] |\psi\rangle|}{2}$$
What is your interpretation of this inequation?

3. $\mathbf{X}$ *and* $\mathbf{Y}$ *are two measurables. What are their outcomes? What does the uncertainty principle tells with these measurables when measured for the quantum state* $|0\rangle$*?*