LECTURE 7
QUANTUM ERROR CORRECTING CODES AND A LITTLE BIT OF CLASSICAL ERROR CORRECTING CODES

Quantum computer science and applications

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Presentation of quantum error correcting codes! But we will start with the classical case

Quantum error correcting code are $\left(\text{roughly}\right)$:

- a clever use of classical codes and $\left(\text{syndrome}\right)$ projective measurements

1. Classical Error Correcting Codes: to be Protected Against Classical Errors
2. A First Quantum Error Correcting Code: Shor's Code
3. Calderbank-Shor-Steane (CSS) Codes
4. Stabilizer Codes
5. Threshold Theorem

Building an efficient quantum computer?

Let's go (good luck. . .)! But it is impossible to build architectures that are **completely isolated from the environment: decoherence** (pure states $\mapsto$ mixed states)

Decoherence ($\longleftrightarrow$ Quantum Noise):

There will be "noise" during computations that will modify the results. . .

- What does the "**noise**" mean in the quantum case?
- How to be "**protected**" against the "noise"? Can we also add redundancy as in the classical case?

$\longrightarrow$ Do the classical computation also suffer of errors during computations?

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$\longrightarrow$ Do the classical computation also suffer of errors during computations?

Yes!

How do we proceed to be protected against errors in classical computations?

In the early age: errors in computation, big issue!

$\longrightarrow$ Read the story of R. Hamming in the Bell labs (1947):

https://en.wikipedia.org/wiki/Richard_Hamming

Classically:

- Resource that we need to protect: the bits 0 and 1
- Errors: bits are flipped $\begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$

Breakthrough: Shannon (1948/1949) gave the foundations to protect classical computations against errors but not only!

Protection against errors in computation $\subsetneq$ Information theory

Protect against errors in the quantum world: a much harder problem!

- **Problem** 1: Not enough to protect $|0\rangle$ and $|1\rangle$, every linear combinations $\alpha |0\rangle + \beta |1\rangle$ must be protected as well
- **Problem** 2: Much richer error model than for classical bits (not only "flip"...)
- **Problem** 3: Impossibility to copy qubits before working on it (no cloning theorem)
- **Problem** 4: Measurements modify the qubits...

To overcome these issues: take a look on how we proceed in the classical case!

CLASSICAL ERROR CORRECTING CODES

Suppose that we send bits across a noisy channel

$$001011 \rightsquigarrow 001111$$

How can the receiver detect that an error occurred and correct it?

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How can the receiver detect that an error occurred and correct it?

Do what you do in your everyday life:

Add redundancy!

An example: spell your name over the phone, send first names!

M like Mike, O like Oscar, R like Romeo, A like Alpha, I like India and N like November

An example: over the phone

M like Mike, O like Oscar, R like Romeo, A like Alpha, etc. . .

- We perform an **encoding** (*i.e.*, adding redundancy),

$$\text{M} \mapsto \text{Mike},\ \text{O} \mapsto \text{Oscar},\ \text{R} \mapsto \text{Romeo},\ \text{A} \mapsto \text{Alpha},\ \text{etc.}\ldots$$

- We send the names across the noisy channel (given by a bad communication over the phone),

$$\text{Mike} \xrightarrow{noise} \text{“ike”},\ \text{Oscar} \xrightarrow{noise} \text{“scar”},\ \text{Romeo} \xrightarrow{noise} \text{“meo”},\ \text{Alpha} \xrightarrow{noise} \text{“ alph”}$$

- The receiver can perform a **decoding**: recovering the first names and then the letters,

$$\text{“ike”} \to \text{Mike} \to \text{M},\ \text{“sca”} \to \text{Oscar} \to \text{O},\ \text{“meo”} \to \text{Romeo} \to \text{R},\ \text{“alph”} \to \text{Alpha} \to \text{A}$$

A naive solution: the 3-bits repetition code

Encode bits as:

$$0 \mapsto 000 \quad \text{and} \quad 1 \mapsto 111$$

Binary Symmetric Channel:

Suppose that bits are independently flipped with probability $p < 1/2$

For instance:

$$000 \rightsquigarrow 010 \text{ with probability } p(1-p)^2,\ 000 \rightsquigarrow 011 \text{ with probability } (1-p)p^2, \text{ etc.} \ldots$$

- Decoding: given $b_1 b_2 b_3$ choose the bit that has the majority

$$010 \mapsto 0 \quad \text{and} \quad 110 \mapsto 1$$

Does the 3-bits repetition code offer a better protection against errors than just sending the bit?

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- **Decoding**: given $b_1 b_2 b_3$ choose the bit that has the majority

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Does the 3-bits repetition code offer a better protection against errors than just sending the bit?

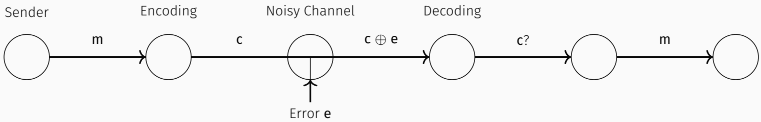
$\longrightarrow$ **Yes**! The probability that choosing the majority bit is the correct choice:

$$3(1-p)^2 p + (1-p)^3 > 1 - p$$

How to transmit k bits over a noisy channel?

1. **Linear code:** fix $\mathcal{C}$ subspace $\subseteq \mathbb{F}_2^n$ of dimension $k < n$
2. **Encoding:** map $(m_1, \ldots, m_k) \longrightarrow \mathbf{c} = (c_1, \ldots, c_n) \in \mathcal{C}$ task adding $n - k$ bits redundancy

   $\longrightarrow$ as $\mathcal{C}$ is linear the encoding is easy $\big($only linear algebra$\big)$
3. Send $\mathbf{c}$ across the noisy channel, bits of $\mathbf{c}$ are independently flipped with probability p



Decoding:

$\longrightarrow$ from $\mathbf{c} \oplus \mathbf{e}$: recover $\mathbf{e}$ and then $\mathbf{c}$ (using the linearity, we easily recover $\mathbf{m}$ from $\mathbf{c}$)

Linear Code:

A linear code $\mathcal{C}$ of length n and dimension k ($[n,k]$-code): subspace of $\mathbb{F}_2^n$ of dimension k

Dual code:

Given $\mathcal{C}$, its dual $\mathcal{C}^\perp$ is the $[n, n-k]$-code

$$\mathcal{C}^\perp \stackrel{\text{def}}{=} \left\{ \mathbf{c}^\perp \in \mathbb{F}_2^n \; : \; \forall \mathbf{c} \in \mathcal{C}, \; \langle \mathbf{c}, \mathbf{c}^\perp \rangle = \sum_{i=1}^{n} c_i c_i^\perp = 0 \in \mathbb{F}_2 \right\}$$

Remark: $\mathcal{C}^\perp$ orthogonal group of $\mathcal{C}$ in the character theory

The repetition code:

The n-repetition code is the following $[n,1]$-code:

$$\left\{ \underbrace{(0,\dots,0)}_{\text{n times}}, \; \underbrace{(1,\dots,1)}_{\text{n times}} \right\}$$

$\longrightarrow$ Using majority voting enables to correct $< n/2$ errors!

But, huge cost of protection: n bits to protect 1 bit. . .

$\mathcal{C}$ is a subspace of $\mathbb{F}_2^n$ of dimension k: choose a basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$ to represent it!

$\longrightarrow$ Many times this representation is not the most "useful"

Parity-check matrix:

Let $\mathbf{h}_1, \ldots, \mathbf{h}_{n-k}$ be a basis of $\mathcal{C}^\perp$, then

$$\mathcal{C} = \left\{ \mathbf{c} \in \mathbb{F}_2^n : \ \mathbf{H}\mathbf{c}^\mathsf{T} = \mathbf{0} \right\} \quad \text{where the rows of } \mathbf{H} \in \mathbb{F}_2^{(n-k)\times n} \text{ are the } \mathbf{h}_i\text{'s}$$

The matrix $\mathbf{H}$ is called a parity-check matrix of $\mathcal{C}$

Given two finite subspaces of $\mathbb{F}_2^n$: $F \subseteq E$.

Equivalence relation: $x \sim y \iff x - y \in F$.

$$E/F = \{\overline{x} \ : \ x \in E\} \quad \text{where} \quad \overline{x} \stackrel{\text{def}}{=} \{y \in E \ : \ x \sim y\} = x + F$$

$\longrightarrow$ It defines a linear space!

$k = \mathbf{dim}\, E/F = \mathbf{dim}\, E - \mathbf{dim}\, F$, in particular: $\sharp E/F = 2^k$

Rough analogy:

E/F	$\mathbb{Z}/4\mathbb{Z}$
$\{\overline{x_1}, \ldots, \overline{x_{2^k}}\}$	$\{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$
$\overline{x_i} = x_i + F$	$\overline{\ell} = \ell + 4\mathbb{Z}$
$\overline{x} = \overline{y} \iff x - y \in F$	$\overline{\ell} = \overline{m} \iff \ell - m \in 4\mathbb{Z}$
$E = \bigsqcup_{1 \leq i \leq 2^k} \overline{x_i}$	$\mathbb{Z} = \bigsqcup_{\ell \in \{0,1,2,3\}} \overline{\ell}$

Decoding: given $\mathbf{c} \oplus \mathbf{e}$, recover $\mathbf{e}$

$\longrightarrow$ Make **modulo $\mathcal{C}$** to extract the information about $\mathbf{e}$

Coset space: $\mathbb{F}_2^n/\mathcal{C}$

$$\sharp\, \mathbb{F}_2^n/\mathcal{C} = 2^{n-k} \quad \text{and} \quad \mathbb{F}_2^n/\mathcal{C} = \left\{ \bar{\mathbf{x}}_i \ : 1 \leq i \leq 2^{n-k} \right\} = \left\{ \mathbf{x}_i + \mathcal{C} \ : \ 1 \leq i \leq 2^{n-k} \right\}$$

where the $\mathbf{x}_i$'s are the **representatives** of $\mathbb{F}_2^n/\mathcal{C}$. The $x_i + \mathcal{C}$'s **are disjoint**!

A natural set of representatives via a parity-check $\mathbf{H}$: **syndromes**

$$\mathbf{x}_i + \mathcal{C} \in \mathbb{F}_2^n/\mathcal{C} \longmapsto \mathbf{H}x_i^{\intercal} \in \mathbb{F}_2^{n-k} \ \left(\text{called a syndrome}\right)$$

is an isomorphism

$$\mathbb{F}_2^n = \bigsqcup_{\mathbf{s} \in \mathbb{F}_2^{n-k}} \left\{ \mathbf{z} \in \mathbb{F}_2^n \ : \ \mathbf{H}\mathbf{z}^{\intercal} = \mathbf{s}^{\intercal} \right\}$$

$\mathbf{c} \oplus \mathbf{e} \bmod \mathcal{C} = \mathbf{H}(\mathbf{c} \oplus \mathbf{e})^{\intercal} = \underbrace{\mathbf{H}\mathbf{c}^{\intercal}}_{=0} \oplus \mathbf{H}\mathbf{e}^{\intercal} = \mathbf{H}\mathbf{e}^{\intercal}$ which gives information to recover $\mathbf{e}$ (decoding)

$\longrightarrow$ $\mathbf{c} \oplus \mathbf{e} \bmod \mathcal{C}$ is only function of $\mathbf{e}$!

Let $\mathcal{C}_{\text{Ham}}$ be the $[7,4]$-code with parity-check matrix:

$$\mathbf{H} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Let $\mathbf{c} \oplus \mathbf{e}$ where $\begin{cases} \mathbf{c} \in \mathcal{C}_{\text{Ham}} \\ \text{only one bit of } \mathbf{e} \text{ is } 1 \end{cases}$: how to easily recover $\mathbf{e}$?

A FIRST EXAMPLE: HAMMING CODE

Let $\mathcal{C}_{\text{Ham}}$ be the $[7,4]$-code with parity-check matrix:

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1. Compute the associated syndrome:

$$\mathbf{H}(\mathbf{c} \oplus \mathbf{e})^{\mathsf{T}} = \mathbf{H}\mathbf{c}^{\mathsf{T}} \oplus \mathbf{H}\mathbf{e}^{\mathsf{T}} = \mathbf{H}\mathbf{e}^{\mathsf{T}}$$

2. $\mathbf{e}$ has only one non-zero bit, $\mathbf{H}\mathbf{e}^{\mathsf{T}}$ is a column of $\mathbf{H}$
3. Columns of $\mathbf{H}$ are the binary representation of $1, 2, \ldots, 7$: $\mathbf{H}\mathbf{e}^{\mathsf{T}}$ gives (in binary) the position where there is an error!

Hamming codes can correct one error!

$\longrightarrow$ There are more clever codes than repetition or Hamming codes... In particular these codes don't seem "good". We will see later a criteria (minimum distance) for "good codes"

- Nice lecture notes by Alain Couvreur (with a focus on algebra):

  http://www.lix.polytechnique.fr/~alain.couvreur/doc_ens/lecture_notes.pdf

- The “bible” of error correcting codes: *The theory of error correcting codes*, F.J. MacWilliams , N.J.A. Sloane (1978)

Error correcting codes have a huge impact in theoretical computer science, cryptography, communications, quantum key distribution (QKD), etc. . .

$\longrightarrow$ Let’s go back to the quantum case!

SHOR’S QUANTUM CODE

Inspired by the classical case: repetition code?

$$\alpha |0\rangle + \beta |1\rangle \longmapsto (\alpha |0\rangle + \beta |1\rangle)^{\otimes 3}$$

But is it possible?

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$$\alpha |0\rangle + \beta |1\rangle \longmapsto (\alpha |0\rangle + \beta |1\rangle)^{\otimes 3}$$

But is it possible?

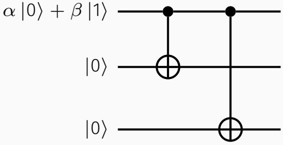
No! No-cloning theorem. . .

Instead consider the following encoding to "mimic the repetition code":

$$(\alpha |0\rangle + \beta |1\rangle) \otimes |00\rangle \longmapsto \alpha |000\rangle + \beta |111\rangle$$

$\longrightarrow$ It is not a repetition code!

To perform encoding, following quantum circuit:



Inspired by the classical case: flip the qubits, i.e. apply X

Error X on the second qubit:

$$\alpha |000\rangle + \beta |111\rangle \rightsquigarrow \alpha |010\rangle + \beta |101\rangle$$

But how to correct this error?

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Error X on the second qubit:

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But how to correct this error?

Use a parity-check matrix!

$\mathbf{H} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ parity-check matrix of the 3-repetition code $\{(000), (111)\}$

$\longrightarrow$ applying to either (010) or (101) gives $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ showing an error occurred to the second bit

Quantumly: implement $|\mathbf{x}\rangle \otimes |00\rangle \mapsto |\mathbf{x}\rangle \otimes |\mathbf{x}\mathbf{H}^{\mathsf{T}}\rangle$ and apply it to

$$(\alpha|010\rangle + \beta|101\rangle) \otimes |00\rangle \longmapsto (\alpha|010\rangle + \beta|101\rangle) \otimes |11\rangle$$

Measure the last two registers and deduce where the X error occurred

$\longrightarrow$ apply **X** on the qubit where there is an error leading to the original quantum state $\left(\mathbf{X}^2 = \mathbf{I}_2\right)$

This method enables to correct any **X** on one qubit

But is it necessary to introduce two ancillary qubits?

Using two auxiliary qubits and **H** was an artefact to mimic the classical case!

$$\alpha |000\rangle + \beta |111\rangle \rightsquigarrow \text{ error?}$$

(*i*) No error,

$$\alpha |000\rangle + \beta |111\rangle \in \mathcal{C}_0 \stackrel{\text{def}}{=} \text{Vect}\left(|000\rangle, |111\rangle\right)$$

If an error **X** occurs we will be in one of the following situations:

(*ii*) First qubit,

$$\alpha |100\rangle + \beta |011\rangle \in \mathcal{C}_1 \stackrel{\text{def}}{=} \text{Vect}\left(|100\rangle, |011\rangle\right)$$

(*iii*) Second qubit,

$$\alpha |010\rangle + \beta |101\rangle \in \mathcal{C}_2 \stackrel{\text{def}}{=} \text{Vect}\left(|010\rangle, |101\rangle\right)$$

(*iv*) Third qubit,

$$\alpha |001\rangle + \beta |110\rangle \in \mathcal{C}_3 \stackrel{\text{def}}{=} \text{Vect}\left(|001\rangle, |110\rangle\right)$$

The $\mathcal{C}_x$'s are the cosets and are orthogonal!

$\longrightarrow$ It defines a measurement: we can decide in which space we live and removing the error

(I) Fundamental idea: decompose the three qubits space as $\left(\text{coset decomposition}\right)$

$$\left(\mathbb{C}^2\right)^{\otimes 3} = \mathcal{C}_0 \overset{\perp}{\oplus} \mathcal{C}_1 \overset{\perp}{\oplus} \mathcal{C}_2 \overset{\perp}{\oplus} \mathcal{C}_3 \tag{1}$$

where,

$$\mathcal{C}_0 \overset{\text{def}}{=} \text{Vect}\left(|000\rangle, |111\rangle\right), \quad \mathcal{C}_1 \overset{\text{def}}{=} \text{Vect}\left(|100\rangle, |011\rangle\right), \quad \mathcal{C}_2 \overset{\text{def}}{=} \text{Vect}\left(|010\rangle, |101\rangle\right)$$

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$$\mathcal{C}_3 \overset{\text{def}}{=} \text{Vect}\left(|001\rangle, |110\rangle\right)$$

$\longrightarrow$ The $\mathcal{C}_x$'s are orthogonal: **it defines a projective measurement!**

(II) Fundamental idea: syndrome measurement

Measure according to Eq. (1). Then apply **X** on a qubit according to the result x. For instance:

$0 \mapsto$ do nothing, $\quad 1 \mapsto$ apply **X** on the first qubit, $\quad 2 \mapsto$ apply **X** on the second qubit, etc

But why does it work?

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But why does it work?

If one error $\mathbf{X}$ occurred, the quantum state will belong with certainty to some $\mathcal{C}_x$ and $\mathbf{X}^2 = \mathbf{I}_2$

Error X on the second qubit:

$$\alpha |000\rangle + \beta |111\rangle \rightsquigarrow \alpha |010\rangle + \beta |101\rangle$$

- Measure according to the orthogonal projections over

$$\mathcal{C}_0 = \text{Vect}(|000\rangle, |111\rangle), \quad \mathcal{C}_1 = \text{Vect}(|100\rangle, |011\rangle), \quad \mathcal{C}_2 = \text{Vect}(|010\rangle, |101\rangle)$$
$$\mathcal{C}_3 = \text{Vect}(|001\rangle, |110\rangle)$$

- With probability one we measure 2 $\left(\text{“we are in } \mathcal{C}_2\text{”}\right)$ and the quantum state does not change

$$\alpha |010\rangle + \beta |101\rangle$$

- Apply **X** on the second qubit

$$\alpha |010\rangle + \beta |101\rangle \longmapsto \alpha |000\rangle + \beta |111\rangle$$

Remarkable fact:

Measurement does not change the quantum state!

Error of type-X on some "random qubit":

$$\alpha |000\rangle + \beta |111\rangle \rightsquigarrow a\,(\alpha |100\rangle + \beta |011\rangle) + b\,(\alpha |010\rangle + \beta |101\rangle) + c\,(\alpha |001\rangle + \beta |110\rangle)$$

Same decoding algorithm: measure according to $\mathcal{C}_0 \overset{\perp}{\oplus} \mathcal{C}_1 \overset{\perp}{\oplus} \mathcal{C}_2 \overset{\perp}{\oplus} \mathcal{C}_3$ but this times the quantum states changes

- With probability $|a|^2$ observe "error on the first qubit", the quantum state collapses to

  $$\alpha |100\rangle + \beta |011\rangle$$

  and apply **X** on the first qubit,
- With probability $|b|^2$ observe "error on the second qubit", the quantum state collapses to

  $$\alpha |010\rangle + \beta |101\rangle$$

  and apply **X** on the second qubit,
- etc. . .

What is the most important sentence of MDC_51002_EP*?*

OTHER KIND OF ERRORS?

What is the most important sentence of MDC_51002_EP*?*

$\longrightarrow$ Quantum computation offers you a huge power with the "-1"

It is the same for errors, errors have a huge power, phase-flip can happen **Z** : $\begin{cases} |0\rangle \mapsto \ \ |0\rangle \\ |1\rangle \mapsto -|1\rangle \end{cases}$

*But is our previous quantum code with its decoding algorithm useful against errors of type-***Z***?*

$\longrightarrow$ No!

Applying Z on some qubit:

$$\alpha\,|000\rangle - \beta\,|111\rangle$$

- Decoding: measuring leads to we are in $\mathcal{C}_0$: "no error" and we do nothing. . .

HOW TO PROTECT AGAINST ERROR OF TYPE-Z?

Fundamental remark:

errors of type $\mathsf{Z} \equiv$ errors of type X in the Fourier basis $|+\rangle, |-\rangle$

$$\mathsf{Z} : \begin{cases} |+\rangle \mapsto |-\rangle \\ |-\rangle \mapsto |+\rangle \end{cases} \quad \text{and} \quad \mathsf{X} : \begin{cases} |+\rangle \mapsto |+\rangle \\ |-\rangle \mapsto -|-\rangle \end{cases}$$

Natural idea: apply $\mathsf{H}^{\otimes 3}$ to $\alpha |000\rangle + \beta |111\rangle$:

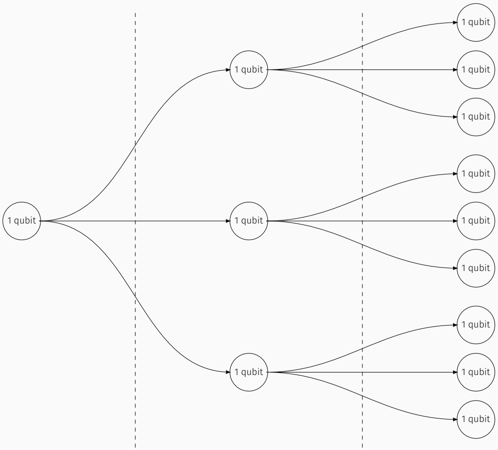
$$\alpha |+++\rangle + \beta |---\rangle$$

As above we can correct any error of type Z on one qubit with this encoding!

$\longrightarrow$ But we are stuck, we cannot correct errors of type-X anymore. . .

Idea: concatenation trick

Encode to protect against **Z**-errors and then encode this to protect against **X**-errors!



Protection against **Z**-errors   Protection against **X**-errors

$$|0\rangle \xrightarrow{\text{1st}} |+++\rangle = \frac{1}{2\sqrt{2}}(|0\rangle + |1\rangle)^{\otimes 3} \xrightarrow{\text{2nd}} \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3}$$

$$|1\rangle \xrightarrow{\text{1st}} |---\rangle = \frac{1}{2\sqrt{2}}(|0\rangle - |1\rangle)^{\otimes 3} \xrightarrow{\text{2nd}} \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}$$

- 1st step: protecting against errors of type-**Z**
- 2nd step: protecting against errors of type-**X**

Encoding:

$$\Big(\alpha |0\rangle + \beta |1\rangle\Big) \otimes \left|0^8\right\rangle \longmapsto \frac{\alpha}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3} + \frac{\beta}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}$$

$$\frac{\alpha}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)^{\otimes 3} + \frac{\beta}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)^{\otimes 3}$$

$\longrightarrow$ The encoding belongs to the linear code of dimension 3 generated by

(111000000), (000111000), (000000111)

As previously, one can define the syndrome measurement according to the cosets:

$$\mathcal{C}_0 \stackrel{\text{def}}{=} \text{Vect}\left(|111000000\rangle, |000111000\rangle, |000000111\rangle\right),$$

$$\mathcal{C}_1 \stackrel{\text{def}}{=} \text{Vect}\left(|011000000\rangle, |100111000\rangle, |100000111\rangle\right), \quad \text{etc}\ldots$$

$\longrightarrow$ 9 subspaces of dimension 3 in orthogonal sum! It defines a (syndrome) measurement

enabling, as previously, to correct any one X-error

Remark:

This syndrome measurement: any interference with any possible Z-error

(change signs not switch vectors of the computational basis)

Once we have removed a possible **X**-error we are left to deal with

$$\frac{\alpha}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)^{\otimes 3} + \frac{\beta}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)^{\otimes 3} = \alpha\,|+_3 +_3 +_3\rangle + \beta\,|-_3 -_3 -_3\rangle$$

$$|+_3\rangle \stackrel{\text{def}}{=} \frac{|000\rangle + |111\rangle}{\sqrt{2}} \quad \text{and} \quad |-_3\rangle \stackrel{\text{def}}{=} \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

$\longrightarrow$ One **Z**-error on any qubit of $|+_3\rangle$ leads to $|-_3\rangle$!

Z-error on either 1st, 2nd or 3rd $\left(\textit{resp.}\ \text{4th, 5th or 6th}\right)$ qubit yields:

$$\alpha\,|-_3 +_3 +_3\rangle + \beta\,|+_3 -_3 -_3\rangle \quad (\textit{resp.}\ \alpha\,|+_3 -_3 +_3\rangle + \beta\,|-_3 +_3 -_3\rangle)$$

- We can define the syndrome measurement: $\left(\mathbb{C}^2\right)^{\otimes 9} = \mathcal{E}_0 \overset{\perp}{\oplus} \mathcal{E}_1 \overset{\perp}{\oplus} \mathcal{E}_2 \overset{\perp}{\oplus} \mathcal{E}_3 \overset{\perp}{\oplus} F$ where:

$$\mathcal{E}_0 \stackrel{\text{def}}{=} \text{Vect}\left(|+_3 +_3 +_3\rangle, |-_3 -_3 -_3\rangle\right),\ \mathcal{E}_1 \stackrel{\text{def}}{=} \text{Vect}\left(|-_3 +_3 +_3\rangle, |+_3 -_3 -_3\rangle\right),\ \dots,\ F \stackrel{\text{def}}{=} \left(\sum_i \mathcal{E}_i\right)^{\perp}$$

Decoding:

Measure $\left(\text{it does not change the quantum state}\right)$ and then apply **Z** on the either the 1st, 2nd or 3rd qubit if the answer is 1, etc. . .

Shor's quantum error correcting code:

It can correct one error of type-$\mathbf{X}$ and one error of type-$\mathbf{Z}$!

Exercise:

Find an error on two qubits which cannot be corrected by Shor's code

- Are the errors of type-$\mathbf{X}$ and $\mathbf{Z}$ be the only possible errors?
- Can Shor's quantum code correct these other potential errors?

$\longrightarrow$ As in classical world: many reasonable models of errors

But there is a moral:

Errors on qubits: apply Pauli matrices

Single qubit Pauli group $\mathcal{P}_1$:

$$\{\pm I_2, \pm X, \pm Y, \pm Z, \pm iI_2, \pm iX, \pm iY, \pm iZ\}$$

$\longrightarrow$ This set **forms a group** for the multiplication!

- $X^2 = Y^2 = Z^2 = I_2$
- The $\neq$ Pauli matrices anti-commute: $XZ = -ZX = -iY$ etc. . .

Exercise Session:

Any 2×2 matrix M on one qubit can be written as:

$$M = e_0 I_2 + e_1 X + e_2 Z + e_3 XZ$$

One reasonable model of error: on each qubit we independently apply a linear operator

Any linear operator **M** on one qubit can be written as:

$$\mathbf{M} = e_0\mathbf{I}_2 + e_1\mathbf{X} + e_2\mathbf{Z} + e_3\mathbf{XZ}$$

$\longrightarrow$ We reduce a continuous set of errors to a discrete set of errors given by **X**, **Z** and **XZ**

Correcting a discrete set of errors by syndrome measurement: **X** and **Z**

$\longrightarrow$ We can automatically correct a much larger $\Big($continuous!$\Big)$ class of errors

Intuitively: if the syndrome measurement is correct with certainty, performing this measurement after applying **M** will collapse the quantum state into no error, error of type-**X** and **Z**

Shor's code can correct all errors of type **X** and **Z**!

Depolarizing channel:

Each qubit independently undergoes an error X, Z or $Y = iXZ$ with probability $p/3$ and is not modified with probability p

On a single qubit, in terms of density operator:

$$\rho \longmapsto \mathcal{E}(\rho) \stackrel{\text{def}}{=} (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$$

$\longrightarrow$ Somehow the quantum analogue of the Binary Symmetric channel

Exercise:

Show that when $p = \frac{3}{4}$, then $\mathcal{E}(\rho) = \frac{I_2}{2}$. How do you interpret this result? What would be the “classical” equivalent with the Binary Symmetric channel?

Quantum channels:

It belongs to a more general theory: quantum measurements, Krauss operators

Errors against which we need to be protected:

X and Z

Decoding Shor's quantum code:

Shor's quantum code can correct any (continuous) error provided they only affect a single qubit

$\longrightarrow$ But to protect one qubit we need nine qubits. . .

Is it useful, namely better than doing nothing?

$\longrightarrow$ Yes! See Exercise Session for a rigorous proof of this statement

(for the depolarizing channel)

Can we do better?

$\longrightarrow$ Yes, let's go! But before break. . .

CSS CODES

We study now Calderbank-Shor-Steane (CSS) codes

Aim:

A more systematic way of encoding quantum states using (classical) linear codes

CSS construction is based on two classical codes:

- the first one corrects errors of type-**X**
- the second one corrects errors of type-**Z**

For any $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_2^n$,

$$\mathsf{X}^{\mathbf{v}} \stackrel{\text{def}}{=} \mathsf{X}^{v_1} \otimes \mathsf{X}^{v_2} \otimes \cdots \otimes \mathsf{X}^{v_n} \quad \text{and} \quad \mathsf{Z}^{\mathbf{v}} \stackrel{\text{def}}{=} \mathsf{Z}^{v_1} \otimes \mathsf{Z}^{v_2} \otimes \cdots \otimes \mathsf{Z}^{v_n}$$

Lemma:

(*i*) $\mathsf{X}^{\mathbf{u}}\mathsf{Z}^{\mathbf{v}} = (-1)^{\langle \mathbf{u},\mathbf{v}\rangle}\mathsf{Z}^{\mathbf{v}}\mathsf{X}^{\mathbf{u}}$

(*ii*) $\mathsf{H}^{\otimes n}\mathsf{X}^{\mathbf{u}} = \mathsf{Z}^{\mathbf{u}}\mathsf{H}^{\otimes n}$ and $\mathsf{H}^{\otimes n}\mathsf{Z}^{\mathbf{u}} = \mathsf{X}^{\mathbf{u}}\mathsf{H}^{\otimes n}$

(*iii*) $\mathsf{Z}^{\mathbf{u}}\,|\mathbf{x}\rangle = (-1)^{\langle \mathbf{u},\mathbf{x}\rangle}\,|\mathbf{x}\rangle$

Proof:

Consequence of the fact that $\mathsf{XZ} = -\mathsf{ZX}$ and $\mathsf{XH} = \mathsf{HZ}$

Lemma:

For any linear code $\mathcal{C}$,

$$\mathsf{H}^{\otimes n}\left|\mathcal{C}\right\rangle = \left|\mathcal{C}^{\perp}\right\rangle \quad \text{where } \left|\mathcal{C}\right\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sharp\mathcal{C}}}\sum_{\mathbf{c}\in\mathcal{C}}\left|\mathbf{c}\right\rangle \quad \text{and} \quad \left|\mathcal{C}^{\perp}\right\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sharp\mathcal{C}^{\perp}}}\sum_{\mathbf{c}^{\perp}\in\mathcal{C}^{\perp}}\left|\mathbf{c}^{\perp}\right\rangle$$

Proof:

See Exercise Session

But from which result this lemma comes from?

Lemma:

For any linear code $\mathcal{C}$,

$$\mathsf{H}^{\otimes n}\,|\mathcal{C}\rangle = \left|\mathcal{C}^{\perp}\right\rangle \quad \text{where} \quad |\mathcal{C}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sharp\mathcal{C}}}\sum_{\mathbf{c}\in\mathcal{C}}|\mathbf{c}\rangle \quad \text{and} \quad \left|\mathcal{C}^{\perp}\right\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sharp\mathcal{C}^{\perp}}}\sum_{\mathbf{c}^{\perp}\in\mathcal{C}^{\perp}}\left|\mathbf{c}^{\perp}\right\rangle$$

Proof:

See Exercise Session

But from which result this lemma comes from?

$\longrightarrow$ Poisson summation formula

- Defined from two linear codes $\left(\mathcal{C}_X, \mathcal{C}_Z\right)$ of length n such that $\mathcal{C}_Z \subseteq \mathcal{C}_X \subseteq \mathbb{F}_2^n$

$$k \stackrel{\text{def}}{=} \dim \mathcal{C}_X / \mathcal{C}_Z = \dim \mathcal{C}_X - \dim \mathcal{C}_Z$$

$\longrightarrow \mathcal{C}_X = \bigsqcup_{1 \leq i \leq 2^k} (\mathbf{x}_i + \mathcal{C}_Z)$ for 2^k vectors $\mathbf{x}_i \in \mathcal{C}_X$ being coset representatives of $\mathcal{C}_X / \mathcal{C}_Z$

There are efficient one-to-one mappings:

$$\mathbf{i} \in \{0,1\}^k \longmapsto \mathbf{x}_i \in \{0,1\}^n \quad \text{and} \quad \mathbf{x}_i \in \{0,1\}^n \longmapsto \mathbf{i} \in \{0,1\}^k$$

CSS quantum codes:

CSS codes encodes k qubits as

$$\sum_{\mathbf{i} \in \{0,1\}^k} \alpha_{\mathbf{i}} \underbrace{|\mathbf{i}\rangle}_{k \text{ qubits}} \otimes \left|0^{n-k}\right\rangle \longmapsto \sum_{\mathbf{x}_i} \alpha_{\mathbf{i}} \underbrace{|\mathbf{x}_i + \mathcal{C}_Z\rangle}_{n \text{ qubits}}$$

where,

$$|\mathbf{x} + \mathcal{C}_Z\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sharp \mathcal{C}_Z}} \sum_{\mathbf{y} \in \mathcal{C}_Z} |\mathbf{x} + \mathbf{y}\rangle$$

Exercise Session:

How to efficiently build CSS encodings?

$\longrightarrow$ As for Shor's code, use: syndrome measurement

Syndrome measurement:

Let $\mathcal{C}$ be a linear code of length n, dimension k and with parity-check matrix $\mathbf{H}$.

We associate to $\mathcal{C}$ and $\mathbf{H}$ the following measurement

$$\left(\mathbb{C}^2\right)^{\otimes n} = \bigoplus_{\mathbf{s}\in\mathbb{F}_2^{n-k}}^{\perp} \mathcal{E}_{\mathbf{s}}^{\mathcal{C}}$$

where,

$$\mathcal{E}_{\mathbf{s}}^{\mathcal{C}} \stackrel{\text{def}}{=} \text{Vect}\left(\underbrace{|\mathbf{z}\rangle}_{n\text{ qubits}} : \mathbf{H}\mathbf{z}^{\mathsf{T}} = \mathbf{s}^{\mathsf{T}}\right) = \text{Vect}\left(|\mathbf{z}\rangle : \mathbf{z} \in \mathbf{x} + \mathcal{C} \text{ where } \mathbf{H}\mathbf{x}^{\mathsf{T}} = \mathbf{s}^{\mathsf{T}}\right)$$

$\longrightarrow$ The $\mathcal{E}_{\mathbf{s}}^{\mathcal{C}}$'s are generated by the vectors of different cosets

But as the cosets are disjoint, the $\mathcal{E}_{\mathbf{s}}^{\mathcal{C}}$'s are orthogonal!

A crucial remark:

If $|\psi\rangle \in \mathcal{E}_{\mathbf{0}}^{\mathcal{C}}$, then $\mathbf{X}^{\mathbf{e}} |\psi\rangle \in \mathcal{E}_{\mathbf{s}}^{\mathcal{C}}$ where $\mathbf{H}\mathbf{e}^{\mathsf{T}} = \mathbf{s}^{\mathsf{T}}$.

$\longrightarrow$ If the $\mathbf{H}\mathbf{e}_i^{\mathsf{T}}$'s are distinct and we can recover $\mathbf{e}_i$ from $\mathbf{H}\mathbf{e}_i^{\mathsf{T}}$: when measuring $\mathbf{X}^{\mathbf{e}_i} |\psi\rangle \in \mathcal{E}_{\mathbf{H}\mathbf{e}_i^{\mathsf{T}}}^{\mathcal{C}}$ we recover $\mathbf{H}\mathbf{e}_i^{\mathsf{T}}$, then $\mathbf{e}_i$ and we can remove $\mathbf{X}^{\mathbf{e}_i}$.

$$\left(|x + \mathcal{C}\rangle = \frac{1}{\sqrt{\sharp\mathcal{C}}} \sum_{c \in \mathcal{C}} |x + c\rangle \right)$$

Starting from the encoding and applying the noise $X^e Z^f$:

$$|\psi\rangle = \sum_{x \in \mathcal{C}_X / \mathcal{C}_Z} \alpha_x |x + \mathcal{C}_Z\rangle \in \mathcal{E}_0^{\mathcal{C}_X} \rightsquigarrow X^e Z^f |\psi\rangle = \sum_{x \in \mathcal{C}_X / \mathcal{C}_Z} \alpha_x X^e Z^f |x + \mathcal{C}_Z\rangle$$

$\longrightarrow$ Z^f only modifies signs! Therefore:

$$\sum_{x \in \mathcal{C}_X / \mathcal{C}_Z} \alpha_x X^e Z^f |x + \mathcal{C}_Z\rangle \in \mathcal{E}_{H_X e^T}^{\mathcal{C}_X}$$ where H_X be a parity-check matrix of $\mathcal{C}_X \supseteq \mathcal{C}_Z$

$\Big($because: $\forall x \in \mathcal{C}_X, c_Z \in \mathcal{C}_Z,\ H_X(x + c_Z)^T = 0$ as $x \in \mathcal{C}_X$ and $c_Z \in \mathcal{C}_Z \subseteq \mathcal{C}_X\Big)$

Syndrome measurement:

It does not modify the quantum state, supposing that we can recover e from $H_X e^T$: remove X^e

$$|\psi\rangle = \sum_{x \in \mathcal{C}_X/\mathcal{C}_Z} \alpha_x |x + \mathcal{C}_Z\rangle \in \mathcal{E}_0^{\mathcal{C}_X} \rightsquigarrow X^e Z^f |\psi\rangle \xrightarrow{\text{1st decoding}} Z^f |\psi\rangle = \sum_{x \in \mathcal{C}_X/\mathcal{C}_Z} \alpha_x Z^f |x + \mathcal{C}_Z\rangle$$

Fundamental remark:

We have the following identities:

$$Z^f |\psi\rangle = \sum_{x \in \mathcal{C}_X/\mathcal{C}_Z} \alpha_x Z^f |x + \mathcal{C}_Z\rangle = \sum_{x \in \mathcal{C}_X/\mathcal{C}_Z} \alpha_x Z^f X^x |\mathcal{C}_Z\rangle$$

By applying $H^{\otimes n}$:

$$\begin{aligned} H^{\otimes n} Z^f |\psi\rangle &= \sum_{x \in \mathcal{C}_X/\mathcal{C}_Z} \alpha_x H^{\otimes n} Z^f X^x |\mathcal{C}_Z\rangle \\ &= \sum_{x \in \mathcal{C}_X/\mathcal{C}_Z} \alpha_x X^f Z^x H^{\otimes n} |\mathcal{C}_Z\rangle \\ &= X^f \sum_{x \in \mathcal{C}_X/\mathcal{C}_Z} Z^x \left|\mathcal{C}_Z^{\perp}\right\rangle \in \text{in the coset given by } \mathbf{H}_Z \mathbf{f}^{\top} \text{ with } \mathbf{H}_Z \text{ parity-check of } \mathcal{C}_Z^{\perp} \end{aligned}$$

Syndrome measurement with $\mathcal{C}_Z^{\perp}$:

Measuring: we can recover $\mathbf{f}$, then we apply $H^{\otimes n}$ leading to $Z^f |\psi\rangle$ and we remove Z^f

Up to now we used the fact that we can "decode" $\mathcal{C}_X$ and $\mathcal{C}_Z^{\perp}$

Let, $\mathbf{H}_X$ and $\mathbf{H}_Z$ be a parity-check matrix of $\mathcal{C}_X$ and $\mathcal{C}_Z^{\perp}$

- To remove errors $\mathbf{X}^{\mathbf{e}_1}$, or $\mathbf{X}^{\mathbf{e}_2}$, . . . , or $\mathbf{X}^{\mathbf{e}_\ell}$:

  the $\mathbf{H}_X\mathbf{e}_i^{\mathsf{T}}$'s have to be distinct and we can efficiently recover $\mathbf{e}_j$ from $\mathbf{H}_X\mathbf{e}_j^{\mathsf{T}}$

- To remove errors $\mathbf{Z}^{\mathbf{f}_1}$, or $\mathbf{Z}^{\mathbf{f}_2}$, . . . , or $\mathbf{Z}^{\mathbf{f}_\ell}$:

  the $\mathbf{H}_Z\mathbf{f}_i^{\mathsf{T}}$'s have to be distinct and we can efficiently recover $\mathbf{f}_j$ from $\mathbf{H}_Z\mathbf{f}_j^{\mathsf{T}}$

But, can we find classical codes offering such "properties"?

$\longrightarrow$ Yes! To understand why it is theoretically possible: minimum distance

Hamming weight:

$$\forall \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{F}_2^n, \quad |\mathbf{x}| \stackrel{\text{def}}{=} \sharp \{ i \in [\![1, n]\!],\ x_i \neq 0 \}$$

Minimum distance:

Let $\mathcal{C} \subseteq \mathbb{F}_2^n$ $\big($linear code$\big)$, its minimum distance is defined as

$$d_{\min}(\mathcal{C}) \stackrel{\text{def}}{=} \min \{ |\mathbf{c}| \ : \ \mathbf{c} \in \mathcal{C} \text{ and } \mathbf{c} \neq \mathbf{0} \}$$

$\longrightarrow$ The minimum distance quantifies how “good” is a code in terms of decoding ability!

Lemma $\big($see Exercise Session$\big)$:

Let $\mathbf{H}$ be any parity-check matrix of $\mathcal{C}$, then

$$\text{the } \mathbf{He}^{\mathsf{T}}\text{'s are distinct } \textbf{when } |\mathbf{e}| < \frac{d_{\min}(\mathcal{C})}{2}$$

$\longrightarrow$ $\mathcal{C}$ can theoretically be decoded if there are $< \frac{d_{\min}(\mathcal{C})}{2}$ errors

Be careful: it does not show the existence of an efficient decoding algorithm, which is far from being guaranteed

- What is the best minimum distance can we expect?

  $\longrightarrow$ It is typically large $\approx n/10$ when $\mathcal{C}$ has dimension $n/2$s $\left(\text{see Exercise Session}\right)$

- Do we know linear codes with a large minimum distance and for which we can remove a large number of errors?

  $\longrightarrow$ Hard question... Yes we can $\left(\text{hopefully for telecommunication}\right)$ but to understand how deserves at least three lectures...

To take away:

It exists codes with a large minimum distance d and we can hope to be able to decode up to $d/2$

But: hard to find codes with a large d and for which we can efficiently decode many errors $\left(\text{even} \ll d/2\right)$

$\longrightarrow$ Active research topic with a lot a consequences, event recent $\left(\text{for instance the 5G}\ldots\right)$

To build CSS codes: choose $\mathcal{C}$ such that (i) can correct many errors and (ii) $\mathcal{C}^{\perp} \subseteq \mathcal{C}$ $\left(\text{weekly auto-dual}\right)$

Theorem: decoding CSS codes

Let $\mathcal{C}_X$ and $\mathcal{C}_Z$ be linear codes such that $\mathcal{C}_Z \subseteq \mathcal{C}_X$

If $\mathbf{e}$ $\left(\textit{resp.}\ \mathbf{f}\right)$ can be recovered from its syndrome by the code $\mathcal{C}_X$ $\left(\textit{resp.}\ \mathcal{C}_Z^{\perp}\right)$, then the quantum error pattern $\mathbf{X}^{\mathbf{e}}\mathbf{Z}^{\mathbf{f}}$ can be corrected by the CSS quantum code associated to the pair $\left(\mathcal{C}_X, \mathcal{C}_Z\right)$

In particular, we can hope to decode up to $d_{\min}(\mathcal{C}_X)/2$ errors-$\mathbf{X}$ and $d_{\min}(\mathcal{C}_Z^{\perp})/2$ errors-$\mathbf{Z}$ $\Big($even combined$\Big)$

See Exercise Session:

- Shor's code $\left(9 \text{ qubits to protect } 1 \text{ qubit}\right)$ is a CSS code
- Steane's code $\left(7 \text{ qubits to protect } 1 \text{ qubit}\right)$ is a CSS code using Hamming codes

STABILIZER CODES

- A class of codes containing CSS codes
- Many similarities with classical linear codes
- Powerful framework for defining/manipulating/constructing/understanding quantum codes

$$
\begin{aligned}
XZ &= -ZX &= -iY \\
XY &= -YX &= iZ \\
YZ &= -ZY &= iX
\end{aligned}
$$

$\longrightarrow$ The elements of $\mathbb{G}_1 = \{\pm 1, \pm i\} \times \{X, Z, Y\}$ commute or anti-commute

$\mathbb{G}_n$-group:

The set of operators of the form $\pm X^{\mathbf{e}} Z^{\mathbf{f}}$ or $\pm i X^{\mathbf{e}} Z^{\mathbf{f}}$, where $\mathbf{e}, \mathbf{f} \in \mathbb{F}_2^n$, form a multiplicative group

Admissible subgroup:

A subgroup $\mathbb{S}$ of $\mathbb{G}_n$ is said to be admissible if: $-\mathbf{I}^{\otimes n} \notin \mathbb{S}$

$\longrightarrow$ We will only consider admissible subgroups!

Lemma:

Any admissible subgroup $\mathbb{S}$ is Abelian (its elements commute)

Proof:

Let $\mathbf{E}, \mathbf{F} \in \mathbb{S} \subseteq \mathbb{G}_n$, then

$$\mathbf{E}^2 = \pm\mathbf{I}, \quad \mathbf{F}^2 = \pm\mathbf{I} \quad \text{and} \quad \mathbf{EF} = \pm\mathbf{FE}$$

But $\mathbf{E}^2, \mathbf{F}^2 \in \mathbb{S}$ and $-\mathbf{I} \notin \mathbb{S}$. Therefore:

$$\mathbf{E}^2 = \mathbf{F}^2 = \mathbf{I}$$

Suppose by contradiction that $\mathbf{EF} = -\mathbf{FE}$, then

$$\mathbf{EFEF} = -\mathbf{EF}^2\mathbf{E} = -\mathbf{I} \in \mathbb{S}\text{: contradiction}$$

Stabilizer code:

$\mathbb{S}$ be an admissible subgroup of $\mathbb{G}_n$

The **stabilizer code** $\mathcal{C}$ associated to $\mathbb{S}$ is defined as

$$\mathcal{C} \stackrel{\text{def}}{=} \left\{ \, |\psi\rangle \, : \, \forall \mathsf{M} \in \mathbb{S}, \; \mathsf{M}\,|\psi\rangle = |\psi\rangle \, \right\}$$

An example:

Vect $(|000\rangle\,, |111\rangle)$ is a stabilizer code associated to

$$\Big\{ \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I}, \; \mathsf{Z} \otimes \mathsf{Z} \otimes \mathsf{I}, \; \mathsf{Z} \otimes \mathsf{I} \otimes \mathsf{Z}, \; \mathsf{I} \otimes \mathsf{Z} \otimes \mathsf{Z} \Big\}$$

INDEPENDENT GENERATORS: MINIMAL SET OF GENERATORS

Given $\mathbb{S}$ an admissible subgroup of $\mathbb{G}_n$:

- **Generators set**: $\mathsf{M}_1, \ldots, \mathsf{M}_\ell$ such that

$$\forall \mathsf{M} \in \mathbb{S},\ \ \mathsf{M} = \mathsf{M}_1^{e_1} \cdots \mathsf{M}_\ell^{e_\ell} \ \text{ for } \ e_1, \ldots, e_\ell \in \{0, 1\}$$

Notation:

$$\langle \mathsf{M}_1, \ldots, \mathsf{M}_\ell \rangle \overset{\text{def}}{=} \left\{ \mathsf{M}_1^{e_1} \cdots \mathsf{M}_\ell^{e_\ell} \ \text{ for } \ e_1, \ldots, e_\ell \in \{0, 1\} \right\}$$

- **Minimal generators set** $\Big($independent generators in the literature$\Big)$: $\mathsf{M}_1, \ldots, \mathsf{M}_\ell$ such that

$$\forall i, \quad \langle \mathsf{M}_1, \ldots, \mathsf{M}_{i-1}, \mathsf{M}_{i+1}, \ldots, \mathsf{M}_\ell \rangle \subsetneq \langle \mathsf{M}_1, \ldots, \mathsf{M}_\ell \rangle$$

Proposition $\Big($admitted$\Big)$:

$\mathbb{S}$ admits a minimal generator set $\mathsf{M}_1, \ldots, \mathsf{M}_r$ for some r and

$$\sharp \mathbb{S} = 2^r.$$

$\mathbb{S} \subseteq \mathbb{G}_n$ admissible subgroup

$\sharp\mathbb{S} = 2^r$ and $\mathbf{M}_1, \ldots, \mathbf{M}_r$ minimal set of generators

The syndrome function:

$$\sigma : \mathbb{G}_n \longrightarrow \{0,1\}^r$$
$$\mathbf{E} \longmapsto \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{pmatrix} \quad \text{with} \quad s_i \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \mathbf{E}\mathbf{M}_i = \mathbf{M}_i\mathbf{E} \\ 1 & \text{if } \mathbf{E}\mathbf{M}_i = -\mathbf{M}_i\mathbf{E} \end{cases}$$

Remark:

For any $\mathbf{M} \in \mathbb{S}$: $\sigma(\mathbf{M}) = \mathbf{0}$

Syndrome: $\sigma(\mathsf{E}) = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{pmatrix}$ with $s_i \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \mathsf{E}\mathsf{M}_i = \mathsf{M}_i\mathsf{E} \\ 1 & \text{if } \mathsf{E}\mathsf{M}_i = -\mathsf{M}_i\mathsf{E} \end{cases}$

$$\mathcal{C}(\mathbf{s}) \stackrel{\text{def}}{=} \left\{ |\psi\rangle\,,\ \forall i,\ \mathsf{M}_i\,|\psi\rangle = (-1)^{s_i}\,|\psi\rangle \right\}$$

$$\longrightarrow \mathcal{C}(\mathbf{0}) = \mathcal{C}$$

Proposition (admitted): a quantum measurement that extracts the syndrome

1. For any $\mathsf{E} \in \mathbb{G}_n$ and any $|\psi\rangle \in \mathcal{C}$:
$$\mathsf{E}\,|\psi\rangle \in \mathcal{C}(\sigma(\mathsf{E}))$$
2. $\left(\mathbb{C}^2\right)^{\otimes n}$ decomposes into the orthogonal direct sum:
$$\left(\mathbb{C}^2\right)^{\otimes n} = \bigoplus_{\mathbf{s}\in\mathbb{F}_2^r}^{\perp} \mathcal{C}(\mathbf{s})$$

$\longrightarrow$ The $\mathcal{C}(\mathbf{s})$'s define a measurement!

Proposition (admitted):

For any $\mathbf{s} \in \mathbb{F}_2^r$, there exists $\mathsf{E} \in \mathbb{G}_n$ such that $\mathbf{s} = \sigma(\mathsf{E})$

We have $\dim_{\mathbb{C}}(\mathcal{C}) = 2^{n-r}$

Linear codes	Stabilizer codes
k bits encoded in n bits	k qubits encoded in n qubits
subspace of dimension k	subspace of dimension 2^k
parity-check matrix $\mathbf{H}$	minimal generators set of $\mathbb{S}$
$r = n - k$ rows, n columns	$r = n - k$ generators
syndrome $\in \{0, 1\}^{n-k}$	syndrome $\in \{0, 1\}^{n-k}$

Error: $\mathsf{E} \in \mathbb{G}_n$

$$|\psi\rangle \in \mathcal{C} \rightsquigarrow \mathsf{E}\,|\psi\rangle \in \mathcal{C}(\sigma(\mathsf{E})) \xrightarrow{\text{measurement}} \mathsf{E}\,|\psi\rangle \text{ with the knowledge of } \sigma(\mathsf{E})$$

- But how to extract **E**?

  $\longrightarrow$ classically

- What are the errors that can be corrected?

  $\longrightarrow$ Subtle question!

Suppose: $|\psi\rangle \rightsquigarrow \mathbf{E}\,|\psi\rangle$ where $\mathbf{E} \in \mathbb{G}_n$

$\longrightarrow$ We want to remove $\mathbf{E}$, *i.e.*, to apply $\mathbf{E}^{-1}$

Decoding process:

We compute $\mathbf{E}' \in \mathbb{G}_n$ such that $\mathbf{E}'\mathbf{E}\,|\psi\rangle \in \mathcal{C} = \mathcal{C}(\mathbf{0})$. In other words,

$$\sigma(\mathbf{E}\mathbf{E}') = \mathbf{0}$$

Is $\mathbf{E}' = \mathbf{E}^{-1}$? Is it necessary?

$\longrightarrow$ We don't need $\mathbf{E} = \mathbf{E}^{-1}$, we only need $\mathbf{E}'\mathbf{E}\,|\psi\rangle = |\psi\rangle$

Suppose: $|\psi\rangle \rightsquigarrow \mathsf{E}\,|\psi\rangle \in \mathcal{C}(0) = \mathcal{C} \xrightarrow{\text{measurement}}$ syndrome **0**, no error. . .

Is it a problem? It depends of **E** . . . Is $\mathsf{E}\,|\psi\rangle = |\psi\rangle$ or not?

We can distinguish two types of error E with syndrome 0:

- **Harmless error** (type-**G** like "Good"): $\mathsf{E} \in \mathbb{S}$, in that case

$$\forall\, |\psi\rangle \in \mathcal{C}, \quad \mathsf{E}\,|\psi\rangle = |\psi\rangle$$

- **Harmful error** (type-**B** like "Bad"): $\mathsf{E} \notin \mathbb{S}$, in that case (proof: use the "minimality" of generators)

$$\exists\, |\psi\rangle \in \mathcal{C}, \quad \mathsf{E}\,|\psi\rangle \neq |\psi\rangle$$

Type-**B** errors: cannot be detected and thus cannot be corrected while it may happen $\mathsf{E}\,|\psi\rangle \neq |\psi\rangle$

To overcome this issue: introduce the **minimum distance**

Remark:

An harmful error **E** verifies by definition $\sigma(\mathsf{E}) = \mathbf{0}$

Recall: if $\mathsf{E} \in \mathbb{G}_n$, then $\mathsf{E} = \mathsf{X}^{\mathbf{e}}\mathsf{Z}^{\mathbf{f}}$ $\left(\text{up to } \times\{\pm 1, \pm i\}\right)$ for some $\mathbf{e}, \mathbf{f} \in \mathbb{F}_2^n$,

Weight Pauli group elements:

For any $\mathsf{E} \in \mathbb{G}_n$, we define its weight as,

$$|\mathsf{E}| \stackrel{\text{def}}{=} \sharp\left\{i : e_i \neq f_i \text{ or } e_i = f_i = 1\right\} = \sharp\left\{\mathsf{X}, \mathsf{Y}, \mathsf{Z} \text{ that appear in } \mathsf{E}\right\}$$

For instance:

$$\left|\mathsf{X}^{(1,0,1,0)}\mathsf{Z}^{(0,0,1,1)}\right| = |\mathsf{X} \otimes \mathsf{I} \otimes \mathsf{XZ} \otimes \mathsf{Z}| = |\mathsf{X} \otimes \mathsf{I} \otimes (-i\mathsf{Y}) \otimes \mathsf{Z}| = 3$$

Admissible subgroup minimum distance:

Given an admissible subgroup $\mathbb{S}$ of $\mathbb{G}_n$, we define its minimum distance as,

$$d \stackrel{\text{def}}{=} \min\left(|\mathsf{E}| : \ \mathsf{E} \text{ error of type } \mathsf{B}\right) = \min\left(|\mathsf{E}| : \ \mathsf{E} \notin \mathbb{S}\right)$$

Exercise:

What is the minimum distance of $\mathrm{Vect}(|000\rangle, |111\rangle)$? Don't forget to exhibit the associated admissible subgroup

Theorem:

$\mathcal{C}$ stabilizer code of minimum distance d, and $|\psi\rangle \in \mathcal{C}$ be corrupted by an error $\mathsf{E} \in \mathbb{G}_n$ of weight $t < d/2$, then $|\psi\rangle$ can be recovered

Proof:

1. $\mathsf{E}\,|\psi\rangle \xrightarrow{\textit{measurement}} \mathsf{E}\,|\psi\rangle$ giving the classical information $\sigma(\mathsf{E})$
2. Find classically minimum weight $\mathsf{E}' \in \mathbb{G}_n$ such that $\sigma(\mathsf{E}') = \sigma(\mathsf{E})$, in particular $|\mathsf{E}'| \leq |\mathsf{E}| = t$

   $\longrightarrow$ We need: efficient classical algorithm coming with the stabiliser group for this task
3. Apply E'. But why does it work?

$$\sigma(\mathsf{E}'\mathsf{E}) = \sigma(\mathsf{E}') + \sigma(\mathsf{E}) = \mathbf{0} \quad \text{and} \quad |\mathsf{E}'\mathsf{E}| \leq |\mathsf{E}'| + |\mathsf{E}| \leq 2t < d$$

   Therefore, by definition of the minimum distance: $\mathsf{E}'\mathsf{E} \in \mathbb{S}$ and $\mathsf{E}'\mathsf{E}\,|\psi\rangle = |\psi\rangle$

- Decoding stabilizer codes:
  - Computing the syndrome by a projective measurement : quantum step
  - Determining the most likely error: classical step
  - Inverting the error: quantum step
- Decoding with certainty up to $d/2$ where $d = \min\left(|\mathbf{E}| \;:\; \mathbf{E} \in \mathbb{G}_n \setminus \mathbb{S}\right)$ $\left(\text{minimum distance}\right)$

  $\longrightarrow$ Be careful: to be efficient, we need to be efficient during the classical step
- We have seen quantum codes $\big($and their decoding algorithm$\big)$:

Shor $\subsetneq$ CSS $\subsetneq$ Stabilizer

See Exercise Session:

- Shor's code $\big($9 qubits to protect 1 qubit$\big)$ is a CSS code
- Steane's code $\big($7 qubits to protect 1 qubit$\big)$ is a CSS code using Hamming codes
- There is a stabilizer code $\big($5 qubits to protect 1 qubit$\big)$ which is not CSS

If you are interested by quantum error correcting codes:

- Kitaev's toric code in the lecture notes, Section 5, by Gilles Zémor
  https://www.math.u-bordeaux.fr/~gzemor/QuantumCodes.pdf

THRESHOLD THEOREM

BUT. . .

I cheated during all this lecture. . .

Why?

I cheated during all this lecture. . .

Why?

Noisy quantum gates?

To encode qubits: use quantum gates. . .

If quantum gates are noisy, then our encodings are not valid and our analysis is false. . .

Do we conclude that quantum codes are only useful with perfect quantum gates?

$\longrightarrow$ **No!** Hopefully. . .

THE THRESHOLD THEOREM

Threshold theorem (admitted, see Nielsen & Chuang):

A quantum circuit containing $p(n)$ gates may be simulated with probability of error at most ε using

$$O\left(\text{poly}\left(\log\left(\frac{p(n)}{\varepsilon}\right)p(n)\right)\right)$$

gates on hardware whose components fail with probability at most p, **if p is below some constant threshold, $p < p_{\text{th}}$**, and given reasonable assumptions about the noise in the hardware.

If the error to perform each gate is a small enough constant: arbitrarily long quantum computations to arbitrarily good precision with small overhead in the number of gates

Proof strategy:

Build recursively from noisy quantum gates better $\left(\text{and larger}\right)$ gates with the help of codes

$\longrightarrow$ The threshold p_{th} depends of the used quantum correcting codes

To take away: Scott Aaronson

" The entire content of the Threshold Theorem is that you're correcting errors faster than they're created. That's the whole point, and the whole non-trivial thing that the theorem shows. That's the problem it solves."

EXERCISE SESSION