LECTURE 6
PHASE ESTIMATION, SHOR'S ALGORITHM AND HIDDEN SUBGROUP PROBLEM

Quantum Information and Computing

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Presentation of Shor's algorithm and hidden Abelian subgroup problem!

It will rely $\left(\text{partly}\right)$ on:

- phase estimation and consequences: **QFT** over finite Abelian groups and order finding

1. Phase Estimation
2. Application 1: Quantum Fourier Transform on $\mathbb{Z}/N\mathbb{Z}$ and any Finite Abelian Group
3. Application 2: Order Finding
4. Shor's Algorithm
5. Hidden Subgroup Problem $\left(\text{HSP}\right)$

PHASE ESTIMATION

Phase estimation:

- **Input:** a unitary **U** and an eigenstate $|u\rangle$:

$$\mathbf{U} |u\rangle = e^{2i\pi\varphi} |u\rangle$$

- **Output:** $\varphi \in [0,1)$, *i.e.*, the knowledge of the associate eigenvalue of $|u\rangle$

$\longrightarrow$ Essential for computing $\mathbf{QFT}_{\mathbb{Z}/N\mathbb{Z}}$ and Shor's algorithm!

Proposition:

We can determine $\left(\text{by using } \mathbf{QFT}_{\mathbb{Z}/2^t\mathbb{Z}}\right)$ the first n bits of φ with probability $1-\varepsilon$ using

$$O(t^2) \text{ elementary gates where } t = n + \left\lceil \log\left(2 + \frac{1}{2\varepsilon}\right)\right\rceil$$

$\longrightarrow$ n bits of φ with probability $1 - e^{-Cn}$ but working in the space of t-qubits with $t = O(n)$

Notation:

Given $j_1, j_2, \ldots, j_m \in \{0, 1\}$:

$$0.j_1j_2\ldots j_m \stackrel{\text{def}}{=} \frac{j_1}{2} + \frac{j_2}{4} + \cdots + \frac{j_m}{2^m} = \sum_{i=1}^{m} \frac{j_i}{2^i}$$

Example:

$$0.101 = \frac{1}{2} + \frac{1}{8} = 0.625, \quad 0.111 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875 \quad \text{and} \quad 0.011 = \frac{1}{4} + \frac{1}{8} = 0.325$$

$$2^m\ 0.j_1j_2\ldots j_m = 2^{m-1}j_1 + 2^{m-2}j_2 + \cdots + j_m = j_1\ldots j_m \in [\![0, 2^m - 1]\!]$$

(binary representation with m bits)

$$2^\ell\ 0.j_1j_2\ldots j_m = \underbrace{2^{\ell-1}j_1 + \cdots + j_\ell}_{\in\mathbb{N}} + 0.j_{\ell+1}\ldots j_m$$

$$\longrightarrow e^{2i\pi 2^\ell \cdot 0.j_1j_2\ldots j_m} = e^{2i\pi 0.j_{\ell+1}\ldots j_m}$$

The quantum algorithm to determine the phase starts from $\left(|u\rangle \text{ being the eigenstate} \right)$

$$\left|0^t\right\rangle |u\rangle$$

$\longrightarrow$ t function of: (i) accuracy and (ii) probability we wish to be successful

Phase estimation, two stages algorithm:

1. Build the following quantum state:

$$\frac{1}{2^{t/2}}\left(|0\rangle + \mathrm{e}^{2i\pi 2^{t-1}\varphi}|1\rangle\right) \otimes \left(|0\rangle + \mathrm{e}^{2i\pi 2^{t-2}\varphi}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + \mathrm{e}^{2i\pi 2^{0}\varphi}|1\rangle\right) \otimes |u\rangle$$

2. Apply the $\mathbf{QFT}^{-1}_{\mathbb{Z}/2^t\mathbb{Z}}$ to reach:

$$\approx \left|\lfloor 2^t\varphi \rfloor\right\rangle \otimes |u\rangle = |\varphi_1 \ldots \varphi_t\rangle \otimes |u\rangle$$

Does the first step remind you of something?

The controlled U^{2^j}-unitary:

$$|1\rangle |u\rangle \longmapsto |1\rangle \mathbf{U}^{2^j} |u\rangle = e^{2i\pi\varphi 2^j} |1\rangle |u\rangle$$
$$|0\rangle |u\rangle \longmapsto |0\rangle |u\rangle$$

Be careful: $\mathbf{U}^{2^j} = \underbrace{\mathbf{U}\cdots\mathbf{U}}_{2^j \text{ iterates}}$, in particular $\mathbf{U}^{2^j} |u\rangle \neq (\mathbf{U}|u\rangle)^{2^j}$

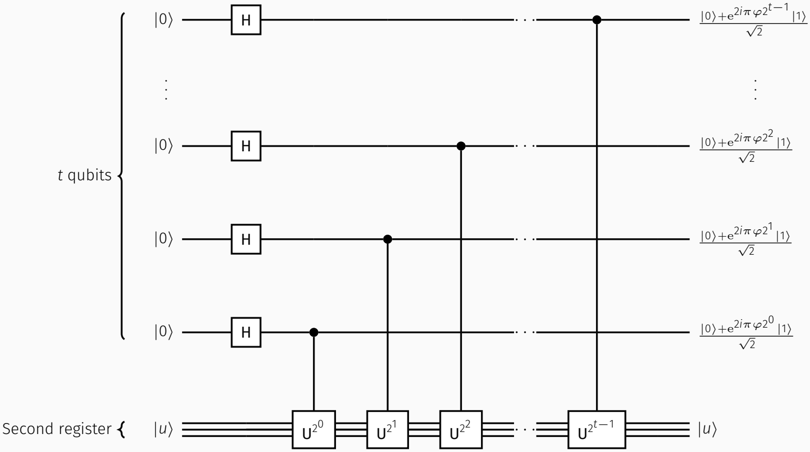
The algorithm:

1. Start with $|0^t\rangle |u\rangle$
2. Apply $\mathbf{H}^{\otimes t} \otimes \mathbf{I}$
3. For $i = 1$ to n:

   apply the controlled $\mathbf{U}^{2^j}$-gate to the i-th register

Resulting quantum state:

$$\frac{1}{2^{t/2}} \left(|0\rangle + e^{2i\pi 2^{t-1}\varphi} |1\rangle\right) \otimes \left(|0\rangle + e^{2i\pi 2^{t-2}\varphi} |1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2i\pi 2^{0}\varphi} |1\rangle\right) \otimes |u\rangle$$



But what is the cost for computing $\mathbf{U}^{2^j}$? Is it 2^j?

BE CAREFUL (I)

Given an arbitrary $\mathbf{U}$, computing the controlled-$\mathbf{U}^{2^j}$ costs $2^j \times \mathbf{Cost(U)} \ldots$

An example:

If $f : \{0,1\}^n \to \{0,1\}^n$ is a bijection efficiently computable, then the unitary

$$\mathbf{U} : |x\rangle \mapsto |f(x)\rangle$$

is efficiently computable. But, is

$$\mathbf{U}^{2^j} : |x\rangle \mapsto \left| f^{2^j}(x) \right\rangle \quad \left(f^{2^j} \text{ composition, not exponentiation} \right)$$

efficiently computable? It depends of the particular shape of $f \ldots$

$\longrightarrow$ Does it imply that phase estimation has an exponential cost?

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efficiently computable? It depends of the particular shape of $f\ldots$

$\longrightarrow$ Does it imply that phase estimation has an exponential cost?

No. . . or Yes. . . It depends!

As in the classical case: computing f^{2^j} is expensive $\left(2^j \times \mathrm{Cost}(f)\right)$ except for some functions. . .

Phase estimation: be careful, in the general case

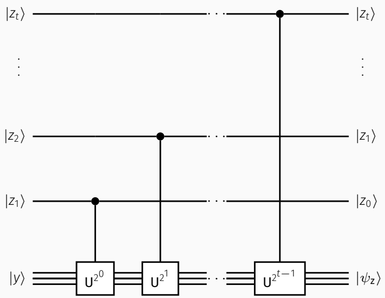
Computing $\mathbf{U}^{2^j}$ costs $2^j \times \mathrm{Cost}(\mathbf{U})$ unless one succeeds to use the particular shape of $\mathbf{U}\ldots$

BE CAREFUL (II)

All the game in phase estimation lies in computing efficiently $\big($designing an efficient circuit$\big)$

$$z_1, \dots, z_t \in \{0,1\}^t, \quad \mathbf{V} : |z_1 \dots z_t\rangle\,|u\rangle \mapsto |z_1 \dots z_t\rangle\,|\psi_{\mathbf{z}}\rangle$$

where **V** is the following unitary



Phase estimation: be careful

Computing $\mathbf{U}^{2^j}$ costs $2^j \times$ **Cost(U)** unless one succeeds to use the particular shape of **U** . . .

$\longrightarrow$ Let us take a look at the classical case!

CLASSICAL EXPONENTIATION: FAST OR TERRIBLY SLOW, CHOOSE!

What is the cost to compute x^{2^j}? Is it 2^j?

Classical exponentiation: fast or terribly slow, choose!

What is the cost to compute x^{2^j}? Is it 2^j?

Of course not... fast exponentiation

- Stupid algorithm: $y = 1$ and then 2^j times: $y \leftarrow yx$; output y
- Clever algorithm: if j even, $y \leftarrow 2^{2^{j/2}}$; outputs y^2; otherwise $y \leftarrow 2^{2^{(j-1)/2}}$ then outputs $2y^2$.

$\longrightarrow$ To compute $2^{2^{j/2}}$ or $2^{2^{(j-1)/2}}$: recursive call

Cost?

- Stupid algorithm: 2^j multiplications!
- Clever algorithm: $\log 2^j = j$ recursive calls and 1 or 2 multiplications for each call

$$\longrightarrow \text{It costs } j \times \underbrace{j^2}_{\text{cost of squaring}}$$

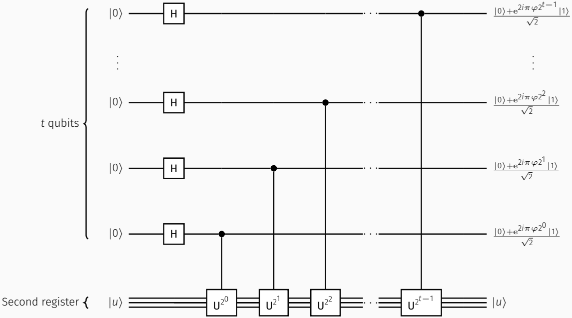
$\longrightarrow$ The "clever" algorithm is exponentially faster...

Be careful: we have used the particular shape of $x \mapsto x^{2^j}$

Usually $f^{2^j}(x) \neq f^{2^{j/2}}(x)^2$ but $f^{2^j}(x) = f^{2^{j/2}}\left(f^{2^{j/2}}(x)\right)$

$$\mathsf{U}\,|u\rangle = e^{2i\pi\varphi}\,|u\rangle \implies \mathsf{U}^{2^j}\,|u\rangle = e^{2i\pi 2^j\varphi}\,|u\rangle$$

$$\text{C-U}^{2^j}\,|0\rangle\,|u\rangle = |0\rangle\,|u\rangle \quad \text{and} \quad \text{C-U}^{2^j}\,|1\rangle\,|u\rangle = e^{2i\pi 2^j\varphi}\,|1\rangle\,|u\rangle$$



- First Step:

$$\frac{1}{\sqrt{2^t}}\left(|0\rangle + |1\rangle\right)^{\otimes t} \otimes |u\rangle$$

- Second Step:

$$\frac{1}{\sqrt{2^t}}\left(|0\rangle + e^{2i\pi 2^{t-1}\varphi}\,|1\rangle\right) \otimes \left(|0\rangle + e^{2i\pi 2^{t-2}\varphi}\,|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2i\pi 2^{0}\varphi}\,|1\rangle\right) \otimes |u\rangle$$

Suppose that

$$\varphi = 0.\varphi_1 \dots \varphi_t$$

See Lecture 5:

$$\frac{1}{2^{t/2}}\left(|0\rangle + e^{2i\pi 2^{t-1}\varphi}\,|1\rangle\right) \otimes \left(|0\rangle + e^{2i\pi 2^{t-2}\varphi}\,|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2i\pi 2^{0}\varphi}\,|1\rangle\right) \otimes |u\rangle$$
$$= \frac{1}{2^{t/2}}\left(|0\rangle + e^{2i\pi 0.\varphi_t}\,|1\rangle\right) \otimes \left(|0\rangle + e^{2i\pi 0.\varphi_{t-1}\varphi_t}\,|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2i\pi 0.\varphi_1\varphi_2\cdots\varphi_t}\,|1\rangle\right) \otimes |u\rangle$$
$$= \mathrm{QFT}_{\mathbb{Z}/2^t\mathbb{Z}}\,|\varphi_1 \dots \varphi_t\rangle$$

Applying $\mathrm{QFT}^{-1}_{\mathbb{Z}/2^t\mathbb{Z}}$ leads to:

$|\varphi_1 \dots \varphi_t\rangle \longrightarrow$ we have recovered φ!

$\longrightarrow$ But what does happen if $\varphi = 0.\varphi_1 \dots \varphi_t\varphi_{t+1}\varphi_{t+2} \dots \varphi_\ell \dots$?

Important convention:

When working in $\mathbb{Z}/2^t\mathbb{Z}$ the considered Hilbert space is $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{t \text{ times}}$ and for all $x \in \mathbb{Z}/2^t\mathbb{Z}$,

$$|x\rangle \stackrel{\text{def}}{=} |x_1 \ldots x_t\rangle$$

where $x_1 \ldots x_t$ being the binary decomposition of x, *i.e.*, $x = \sum_{k=1}^{t} x_k 2^{t-k}$

$$\frac{1}{2^{t/2}} \left(|0\rangle + e^{2i\pi 2^{t-1}\varphi} |1\rangle\right) \otimes \left(|0\rangle + e^{2i\pi 2^{t-2}\varphi} |1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2i\pi 2^{0}\varphi} |1\rangle\right) \otimes |u\rangle$$

$$= \frac{1}{2^{t/2}} \sum_{\ell=0}^{2^t-1} e^{2i\pi \ell \varphi} |\ell\rangle \otimes |u\rangle$$

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$$\frac{1}{2^{t/2}} \left(|0\rangle + e^{2i\pi 2^{t-1}\varphi} |1\rangle \right) \otimes \left(|0\rangle + e^{2i\pi 2^{t-2}\varphi} |1\rangle \right) \otimes \cdots \otimes \left(|0\rangle + e^{2i\pi 2^{0}\varphi} |1\rangle \right) \otimes |u\rangle$$

$$= \frac{1}{2^{t/2}} \sum_{\ell=0}^{2^t-1} e^{2i\pi \ell \varphi} |\ell\rangle \otimes |u\rangle$$

Applying $\mathrm{QFT}^{-1}_{\mathbb{Z}/2^t\mathbb{Z}} \otimes \mathrm{Id}$ leads to:

$$\mathrm{QFT}^{-1}_{\mathbb{Z}/2^t\mathbb{Z}} \otimes \mathrm{Id} \left(\frac{1}{2^{t/2}} \sum_{\ell=0}^{2^t-1} e^{2i\pi \ell \varphi} |\ell\rangle \otimes |u\rangle \right) = \frac{1}{2^t} \sum_{k,\ell=0}^{2^t} e^{2i\pi \ell (\varphi - \frac{k}{2^t})} |k\rangle \otimes |u\rangle$$

Best approximation of φ for the first t bits:

Le $b \in [\![0, 2^t - 1]\!]$ be such that $b/2^t = 0.b_1 \ldots b_t$ and

$$0 \leq \varphi - \frac{b}{2^t} \leq 2^{-t} \quad : b/2^t \text{ best } t \text{ bits approximation of } \varphi$$

Up to now we have the following quantum state:

$$\frac{1}{2^t} \sum_{k,\ell=0}^{2^t-1} \mathrm{e}^{2i\pi\ell(\varphi - \frac{k}{2^t})} |k\rangle |u\rangle$$

Let α_j be the amplitude of $(b + j \bmod 2^t)$ in the first register:

$$|\alpha_j|^2 = \frac{1}{2^{2t}} \left| \sum_{\ell=0}^{2^t-1} \left(\mathrm{e}^{2i\pi\left(\varphi - \frac{b+j}{2^t}\right)} \right)^k \right|^2$$

Measure (see Exercise Session):

Let $m \in \{0,1\}^t$ be the outcome after measuring the first register in the computational basis $\left(\text{defining an integer in } [\![0, 2^t - 1]\!]\right)$. We have

$$\mathbb{P}\left(|b - m| > \alpha\right) \leq \frac{1}{2(\alpha - 1)}$$

Best approximation of φ for the first t bits:

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$\longrightarrow$ Determining φ with n bits of accuracy thanks to the output of the measure m $\left(t > n\right)$:

$$\left|\frac{b}{2^t} - \frac{m}{2^t}\right| < 2^{-n}$$

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$$\left|\frac{b}{2^t} - \frac{m}{2^t}\right| < 2^{-n}$$

$\longrightarrow$ Therefore: choosing $\alpha = 2^{t-n} - 1$ in the above probability...

But to reach a probability of success $\geq 1 - \varepsilon$:

$$\frac{1}{2(\alpha - 1)} = \frac{1}{2(2^{t-n} - 2)} \leq \varepsilon \iff t = n + \left\lceil \log\left(2 + \frac{1}{2\varepsilon}\right)\right\rceil$$

Phase estimation:

- **Input:** a unitary $\mathbf{U}$ and an eigenstate $|u\rangle$:

$$\mathbf{U}|u\rangle = e^{2i\pi\varphi}|u\rangle$$

- **Output:** $\varphi \in [0, 1)$, *i.e.*, the knowledge of the associate eigenvalue of $|u\rangle$

Proposition:

The phase estimation $\Big($before the last step measuring in the computational basis$\Big)$ computes,

$$|0^t\rangle|u\rangle \mapsto |\psi_u\rangle|u\rangle$$

such that $|\psi_u\rangle$ is an approximation of φ, *i.e.* when measuring the first register we obtain $\widetilde{\varphi} \in \{0,1\}^t$ admitting the same first n bits than φ with probability $\geq 1-\varepsilon$ if t is chosen as

$$t = n + \left\lceil \log\left(2 + \frac{1}{2\varepsilon}\right)\right\rceil$$

Furthermore, the algorithm uses $O(t^2)$ elementary gates and t calls to controlled-$\mathbf{U}^{2^j}$ for $0 \leq j < t$

- Be careful: we need to compute $\left(U^{2^j}\right)_{0\leq j\leq t}$ which has a cost $\geq 2^t$ unless one uses the particular shape of U. . .
- Accuracy with a probability exponentially close to 1 at the cost of a "constant" overhead: n bits of φ with probability $1 - \mathrm{e}^{-Cn}$ but with $t = O(n)$
- Be careful: to run phase estimation we also need to be able to compute the eigenvector $|u\rangle$ . . .

APPLICATION 1: QFT OVER $\mathbb{Z}/N\mathbb{Z}$

Recall that characters of

- $\mathbb{F}_2^n$: $\chi_{\mathsf{x}}(\mathsf{y}) = (-1)^{\mathsf{x}\cdot\mathsf{y}}$
- $\mathbb{Z}/2^n\mathbb{Z}$: $\chi_x(y) = \mathrm{e}^{\frac{2i\pi xy}{2^n}}$
- $\mathbb{Z}/N\mathbb{Z}$: $\chi_x(y) = \mathrm{e}^{\frac{2i\pi xy}{N}}$

Lecture 5: computing efficiently $\big(O(n)$ and $O(n^2)\big)$

$$\mathrm{QFT}_{\mathbb{F}_2^n} = \mathsf{H}^{\otimes n} : |\mathsf{x}\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{\mathsf{y}\in\{0,1\}^n} (-1)^{\mathsf{x}\cdot\mathsf{y}} |\mathsf{y}\rangle \quad \text{and} \quad \mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}} : |x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{y\in\mathbb{Z}/2^n\mathbb{Z}} \mathrm{e}^{\frac{2i\pi xy}{2^n}} |y\rangle$$

Aim: computing efficiently $\mathrm{QFT}_{\mathbb{Z}/N\mathbb{Z}}$ $\big($when N not a power of $2\big)$

$$\mathrm{QFT}_{\mathbb{Z}/N\mathbb{Z}} : |x\rangle \longmapsto \frac{1}{\sqrt{N}} \sum_{y\in\mathbb{Z}/N\mathbb{Z}} \mathrm{e}^{\frac{2i\pi xy}{N}} |y\rangle$$

HOW TO PROCEED?

Computing **QFT**$_{\mathbb{Z}/N\mathbb{Z}}$: use phase estimation!

$$\mathsf{U}_1\Big(\,|k\rangle\,|0\rangle\,\Big) \mapsto |k\rangle\,\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle \quad \text{and} \quad \mathsf{U}_2\Big(\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\,|0\rangle\,\Big) \mapsto \mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\,|k\rangle$$

$\longrightarrow$ These two unitaries are enough to compute $\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle$!

We can perform $\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}$ as:

$$|k\rangle\,|0\rangle \xrightarrow{\mathsf{U}_1} |k\rangle\,\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle \xrightarrow{\mathsf{SWAP}} \mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\,|k\rangle \xrightarrow{\mathsf{U}_2^{-1}} \mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\,|0\rangle$$

$$\mathrm{U}_1\Big(\,|k\rangle\,|0\rangle\,\Big) \mapsto |k\rangle\, \mathrm{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle \quad \text{and} \quad \mathrm{U}_2\Big(\mathrm{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\,|0\rangle\,\Big) \mapsto \mathrm{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\,|k\rangle$$

Be careful: $|k\rangle$ here is such that $k \in \mathbb{Z}/N\mathbb{Z}$ and N may not be a power of two... In particular $|k\rangle$ cannot be written as $|0010\ldots1\rangle$

$\Big(\,|k\rangle\,\Big)_{k\in\mathbb{Z}/N\mathbb{Z}}$ is an orthonormal basis of an Hilbert space of dimension N

$\longrightarrow$ This quantum space is called the space of **qudits**!

Two possibilities to perform computation with qudits: (i) encode qudits in qubits or (ii) implement your quantum device directly with Hilbert spaces of dimension > 2

It is the same issue with classical computer! How to implement trits, namely $\mathbb{Z}/3\mathbb{Z}$?

To build the unitary $|k\rangle\,|0\rangle \mapsto |k\rangle\,\mathbf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle$ (admitting we can perform efficiently the different unitaries over qudits)

1. Start from $|k\rangle\,|0\rangle\,|0\rangle$
2. Apply the "uniform superposition" over the second register
$$|k\rangle\,\frac{1}{\sqrt{N}}\sum_{j\in\mathbb{Z}/N\mathbb{Z}}|j\rangle\,|0\rangle$$
3. Apply the multiplication operator $\left(|x\rangle\,|y\rangle\,|0\rangle \mapsto |x\rangle\,|y\rangle\,|xy \bmod N\rangle\right)$
$$|k\rangle\,\frac{1}{\sqrt{N}}\sum_{j\in\mathbb{Z}/N\mathbb{Z}}|j\rangle\,|kj \bmod N\rangle$$
4. Apply the "phase flip in $\mathbb{Z}/N\mathbb{Z}$" $\left(|x\rangle \mapsto e^{2i\pi\frac{x}{N}}\,|x\rangle\right)$ on the third register
$$|k\rangle\,\frac{1}{\sqrt{N}}\sum_{j\in\mathbb{Z}/N\mathbb{Z}}e^{2i\pi\frac{kj}{N}}\,|j\rangle\,|kj \bmod N\rangle$$
5. Apply the inverse of the multiplication operation:
$$|k\rangle\,\frac{1}{\sqrt{N}}\sum_{j\in\mathbb{Z}/N\mathbb{Z}}e^{2i\pi\frac{kj}{N}}\,|j\rangle\,|0\rangle = |k\rangle\,\mathbf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\,|0\rangle$$

$$\mathsf{U} : |k\rangle \mapsto |k+1 \bmod N\rangle$$

$\longrightarrow \mathsf{U}^{2^j} : |k\rangle \mapsto \left|k + 2^j \bmod N\right\rangle$ can be built in time $O(\log N)$

$\left(x \mapsto x + 2^j \bmod N \text{ can be classically computed in time } O(\log N) \right)$

We have the following computation:

$$\begin{aligned} \mathsf{U}\left(\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle\right) &= \frac{1}{\sqrt{N}} \sum_{y\in\mathbb{Z}/N\mathbb{Z}} \mathrm{e}^{\frac{2i\pi ky}{N}}\,\mathsf{U}|y\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{y\in\mathbb{Z}/N\mathbb{Z}} \mathrm{e}^{\frac{2i\pi ky}{N}}\,|y+1\rangle \\ &= \mathrm{e}^{\frac{-2i\pi k}{N}}\,\frac{1}{\sqrt{N}} \sum_{y\in\mathbb{Z}/N\mathbb{Z}} \mathrm{e}^{\frac{2i\pi ky}{N}}\,|y\rangle \\ &= \mathrm{e}^{2i\pi\frac{N-k}{N}}\,\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle \end{aligned}$$

$\longrightarrow \mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\,|k\rangle$ is an eigenvector of U with eigenvalue $\mathrm{e}^{2i\pi\varphi}$ where $\varphi \overset{\text{def}}{=} \frac{N-k}{N}$

(remember: Fourier basis is the basis where translation operator is diagonal)

COMPUTING THE SECOND UNITARY U_2: USE PHASE AMPLIFICATION

The translation operator **U** *is diagonal in the Fourier basis*

$\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}|k\rangle$ is an eigenvector of **U** with eigenvalue $e^{2i\pi\varphi}$ where

$$\varphi \overset{\text{def}}{=} \frac{N-k}{N}$$

Applying phase estimation with $n = \lceil \log N \rceil$ $\left(\text{bits of precision}\right)$ enables to compute:

$$\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\Big(|k\rangle\Big)|0\rangle \longmapsto \mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}|k\rangle|N-k\rangle$$

$\longrightarrow$ Be careful: phase estimation gives only an approximation of the transform!

Therefore: after applying the unitary $|x\rangle \mapsto |N-x\rangle$ we obtain an approximation of

$$\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\Big(|k\rangle\Big)|0\rangle \longmapsto \mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}\Big(|k\rangle\Big)|k\rangle$$

Cost:

Given $t = O(\log N)$, we have a cost of $O(t^2)$ plus the cost to run $\mathbf{U}^{2^j}: |k\rangle \mapsto \left|k + 2^j \bmod N\right\rangle$ for $0 \leq j < t$ which can be done in time $O(t^3)$ $\left(\text{clever combination of the } \mathbf{U}^{2^j}\text{-controlled}\right)$

$\longrightarrow$ Final cost to compute $\mathsf{QFT}_{\mathbb{Z}/N\mathbb{Z}}$: $O\left(\log^3 N\right)$

WHAT ABOUT THE GENERAL CASE?

Is it possible to efficiently build **QFT**$_G$ where G is any arbitrary finite abelian group?

Is it possible to efficiently build $\mathbf{QFT}_G$ where G is any arbitrary finite abelian group?

$\longrightarrow$ Yes!

How to proceed $\big($**rough explanation**$\big)$:

Any finite abelian group G of size N is isomorphic to the product of cyclic groups:

$$\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$$

Then $\big($admitted$\big)$,

$$\mathbf{QFT}_G \text{ can be written as } \mathbf{QFT}_{\mathbb{Z}/n_1\mathbb{Z}} \otimes \cdots \otimes \mathbf{QFT}_{\mathbb{Z}/n_k\mathbb{Z}}$$

$\longrightarrow$ We deduce that $\mathbf{QFT}_G$ can be computed in time $O\left(\log^3 \sharp G\right)$

Be careful, given a finite Abelian group it is classically hard to compute its decomposition as cyclic groups... Quantum case: end of the lecture

APPLICATION 2: ORDER FINDING

Order finding problem:

- **Input:** integers x, N where $\mathbf{gcd}(x, N) = 1$
- **Output:** least positive integer r such that $x^r = 1 \bmod N$

Solving the factorization reduces to this problem

$$\left(\text{solving order finding} \implies \text{solving factorization}\right)$$

Proposition:

We can quantumly determine the order r $\left(\text{with high probability}\right)$ in time

$$O\left(\log^3 N\right)$$

$\longrightarrow$ Best classical algorithms are sub-exponential in N:

$$\exp\left((c + o(1)) \log^{\alpha}(N) \log^{1-\alpha}(\log N)\right)$$

where c, α are constants

Suppose that we work in the space of L qubits

Given $y \in [\![0, 2^L - 1]\!]$, we will naturally identify $|y\rangle$ to $|y_1 \dots y_L\rangle$

where $y_1 \dots y_L$ binary decomposition of y

For instance:

Given $3, 5 \in [\![0, 2^3 - 1]\!]$,

$$|3\rangle = |011\rangle \quad \text{and} \ |5\rangle = |101\rangle$$

x integer : $\mathbf{gcd}(x, N) = 1$ and r its order, smallest positive integer such that $x^r = 1 \ \mathbf{mod}\ N$

Phase estimation applied to the following unitary and eigenvector:

Let,

$$L \overset{\text{def}}{=} \lceil \log N \rceil \quad \Big(\text{work in the space of } L\text{-qubits}\Big)$$

$$\forall y \in \{0,1\}^L = [\![0, 2^L - 1]\!], \quad \mathbf{U}\,|y\rangle \overset{\text{def}}{=} \begin{cases} |xy \ \mathbf{mod}\ N\rangle & \text{if } 0 \leq y \leq N-1 \\ |y\rangle & \text{otherwise } \Big(N - 1 < y < 2^{\lceil \log N \rceil}\Big) \end{cases}$$

$$\forall s \in [\![0, r]\!], \quad |u_s\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \mathrm{e}^{-\frac{2i\pi sk}{r}} \left| x^k \ \mathbf{mod}\ N \right\rangle \ \text{eigenvector of } \mathbf{U} \text{ with eigenvalue } \mathrm{e}^{2i\pi \frac{s}{r}}$$

$\longrightarrow$ We work here in the space of qubits $\Big($natural trick, identity if integers $\geq N - 1\Big)$

$$\begin{aligned} \mathbf{U}\,|u_s\rangle &= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \mathrm{e}^{-\frac{2i\pi sk}{r}}\, \mathbf{U} \left| x^k \ \mathbf{mod}\ N \right\rangle \\ &= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \mathrm{e}^{-\frac{2i\pi sk}{r}} \left| x^{k+1} \ \mathbf{mod}\ N \right\rangle \\ &= \mathrm{e}^{2i\pi \frac{s}{r}}\, |u_s\rangle \end{aligned}$$

$\longrightarrow$ Be careful: in the last equality we used: x **has order** r **modulo** N**, thus** $x^r = 1 \ \mathbf{mod}\ N$

BUT TWO QUESTIONS

For the eigenvalue $\frac{s}{r}$: we work in $\mathbb{Z}/N\mathbb{Z}$ and with $L = \lceil \log N \rceil$ qubits

To perform efficiently phase estimation, two issues:

- How to compute efficiently the $\mathbf{U}^{2^j}$'s?
- How to compute the eigenvector $|u_s\rangle$?

$\longrightarrow$ We will be able to recover approximations of $\frac{s}{r}$, **not** r...

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- How to compute efficiently the $\mathbf{U}^{2^j}$'s?
- How to compute the eigenvector $|u_s\rangle$?

$\longrightarrow$ We will be able to recover approximations of $\frac{s}{r}$, **not** r...

Be patient!

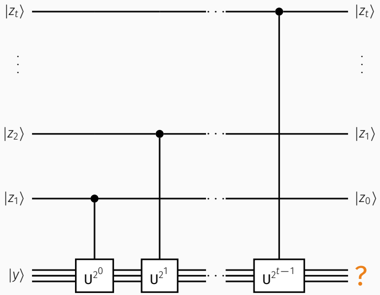
Parameter of phase estimation:

We will determine the first $2L + 1$ bits of $\frac{s}{r}$ with probability $1 - \varepsilon$

$\longrightarrow$ Choose in phase estimation $t = 2L + 1 + \lceil \log\left(2 + \frac{1}{2\varepsilon}\right) \rceil$ qubits

In particular: $t = O(L)$ even if $\varepsilon = e^{-CL}$ with $C > 0$ $\big($constant$\big)$

$$U|y\rangle = |xy \text{ mod } N\rangle \quad (0 \le y \le N-1)$$



The above circuit (used in the phase estimate) performs the following computation:

$$\begin{aligned} |z_t \dots z_1\rangle\,|y\rangle &\longrightarrow |z\rangle\, U^{z_t 2^{t-1}} \cdots U^{z_1 2^0}\,|y\rangle \\ &= |z\rangle \left| x^{z_t 2^{t-1}} \times \cdots \times x^{z_1 2^0} y \text{ mod } N \right\rangle \\ &= |z\rangle\,|yx^z \text{ mod } N\rangle \end{aligned}$$

$\longrightarrow$ To perform **efficiently** phase estimation: compute $|z\rangle\,|y\rangle \mapsto |z\rangle\,|yx^z \text{ mod } N\rangle$ efficiently

$\Big($modular exponentiation$\Big)$

Aim: computing efficiently

$$|z\rangle\,|y\rangle \mapsto |z\rangle\,\left|yx^z \bmod N\right\rangle$$

1. Let $U_{EM} : |z\rangle\,|y\rangle \mapsto |z\rangle\,|y \oplus (x^z \bmod N)\rangle$ $\left(\text{be careful } z \mapsto x^z \bmod N \text{ not bijective}\right)$

$$|z\rangle\,|y\rangle\,|0\rangle \xrightarrow{U_{EM}} |z\rangle\,|y\rangle\,\left|x^z \bmod N\right\rangle \xrightarrow{\text{mult}} |z\rangle\,\left|yx^z \bmod N\right\rangle\left|x^z \bmod N\right\rangle \xrightarrow{U_{EM}^{-1}} |z\rangle\,\left|yx^z \bmod N\right\rangle|0\rangle$$

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2. Computing efficiently $\mathbf{U}_{\mathrm{EM}}$: classically

   $$x \mapsto x^z \bmod N$$

   can by computed in $O\left(\log z\right) = O(\log t) = O\left(\log N\right)$ squaring, therefore $O\left(\log^3 N\right)$ operations

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Conclusion: using phase estimation

We determine the first $2L+1$ bits of $\frac{s}{r}$ with probability $1 - \mathrm{e}^{-CL}$ in time $O(L^3)$ where $L = \lceil \log N \rceil$

Aim: computing

$$|u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \mathrm{e}^{-\frac{2i\pi sk}{r}} \left| x^k \bmod N \right\rangle$$

But we do not know r . . . Our aim is to find it!

The trick:

$$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle = |1\rangle$$

$\longrightarrow$ Plugging $|1\rangle$ in the phase estimation algorithm will give the first $2L+1$ bits of $\frac{s}{r}$ for some

$\Big($uniform and unknown$\Big)$ $s \in [\![0, r-1]\!]$ with probability $1-\varepsilon$

Exercise Session:

Proof of this statement

Up to now we have recovered $\Big($with high probability$\Big)$ in quantum time $O(L^3)$ the first $2L+1$ bits of

$$\frac{s}{r} \quad \text{where} \quad 0 \leq s < r \quad \text{and} \quad s \in [\![1, N-1]\!]$$

$\longrightarrow$ It does not give r, even $\frac{s}{r}$ ...

Theorem $\left(\text{admitted}\right)$ about continued fractions:

Let $\widetilde{\varphi}$ be a rational given as input, let s and r be L bits integers such that

$$\left|\frac{s}{r} - \widetilde{\varphi}\right| < \frac{1}{2r^2}$$

Then, there exists an algorithm $\Big($using "continued fractions"$\Big)$ that outputs (s', r') which verifies

$$\gcd(s', r') = 1 \quad \text{and} \quad \frac{s'}{r'} = \frac{s}{r}$$

using $O(L^3)$ classical operations

In our case:

With probability $1 - \varepsilon$: phase estimation outputs $\widetilde{\varphi}$ an approximation of $\frac{s}{r}$ accurate to $2L + 1$ bits, therefore:

$$\left|\frac{s}{r} - \widetilde{\varphi}\right| \leq \frac{1}{2^{2L+1}} \leq \frac{1}{2r^2} \quad \left(\text{as } r \leq N - 1 \leq L = \lceil \log N \rceil\right)$$

$\longrightarrow$ In time $O(L^3)$ we compute s', r' co-prime such that $\frac{s'}{r'} = \frac{s}{r}$

s', r' are co-prime such that $\frac{s'}{r'} = \frac{s}{r}$

$\longrightarrow$ If $\mathbf{gcd}(s, r) > 1$ then $r' \neq r$, only $r' \mid r \ldots$

A solution $\Big($but inefficient. . . $\Big)$:

The number of prime numbers $< r$ is $\approx r/\log(r)$

$$\longrightarrow \mathbb{P}(\mathbf{gcd}(s, r) = 1) \approx \log(r)/r \text{ as } s \text{ is uniformly picked in } [\![0, r-1]\!]$$

Therefore we need to repeat $\approx r = O(L)$ number of times the algorithm before reaching $\mathbf{gcd}(s, r) = 1$. It will increase the cost from $O(L^3)$ to $O(L^4)$.

Fundamental remark:

$$\frac{s_1'}{r_1'} = \frac{s_1}{r} \implies r\, s_1' = s_1\, r_1' \quad \text{and} \quad \frac{s_2'}{r_2'} = \frac{s_2}{r} \implies r\, s_2' = s_2\, r_2'$$

We have $\mathbf{gcd}(s_1', r_1') = \mathbf{gcd}(s_2', r_2') = 1$, supposing that $\mathbf{gcd}(s_1', s_2') = 1$ implies that $r = \mathbf{lcm}(r_1', r_2')$

$\longrightarrow$ Therefore: obtaining two estimations (s_1', r_1') and (s_2', r_2') and supposing that $\mathbf{gcd}(s_1', s_2') = 1$

we can recover $r = \mathbf{lcm}(r_1', r_2')$.

What is the probability that $\mathbf{gcd}(s_1', s_2') = 1$ given that s_1' and that s_2' are uniformly distributed in $[\![0, r-1]\!]$?

Fundamental remark:

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$\longrightarrow$ Therefore: obtaining two estimations (s_1', r_1') and (s_2', r_2') and supposing that $\mathbf{gcd}(s_1', s_2') = 1$ we can recover $r = \mathbf{lcm}(r_1', r_2')$.

What is the probability that $\mathbf{gcd}(s_1', s_2') = 1$ given that s_1' and that s_2' are uniformly distributed in $[\![0, r-1]\!]$?

$\longrightarrow$ It is $\geq \frac{1}{4}$ (see Exercise Session)

In conclusion:

Repeating the algorithm a constant number of times enables to recover r with probability exponentially close to one (times $(1-\varepsilon)$)!

ORDER FINDING ALGORITHM

To compute the order r of $x \bmod N$ $\left(\text{where } \gcd(x,N)=1\right)$ we first run a constant number of times the phase estimation with $t = 2\lceil \log N \rceil + 1 + \log\left(2 + \frac{1}{2\varepsilon}\right)$. It has been necessary to compute:

- $\mathrm{QFT}_{\mathbb{Z}/2^t\mathbb{Z}}$: done in time $O(t^2) = O(\log^2 N)$
- modular exponentiation $|z\rangle\,|y\rangle \mapsto |z\rangle\,|yx^z \bmod N\rangle$: done in time $O(\log^3 N)$

It outputs a $2\lceil \log N \rceil + 1$ approximation of some $\frac{s}{r}$ where $s \in [\![0, r-1]\!]$ is uniform and unknown

To compute the order r of x $\mathbf{mod}$ N $\left(\text{where } \mathbf{gcd}(x, N) = 1\right)$ we first run a constant number of times the phase estimation with $t = 2\lceil \log N \rceil + 1 + \log\left(2 + \frac{1}{2\varepsilon}\right)$. It has been necessary to compute:

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It outputs a $2\lceil \log N \rceil + 1$ approximation of some $\frac{s}{r}$ where $s \in [\![0, r-1]\!]$ is uniform and unknown

Then after collecting some approximations of $\frac{s_i}{r}$, apply continued fraction algorithm to obtain (s_i', r_i') in time $O\left(\log^3 N\right)$ with $\frac{s_i'}{r_i'} = \frac{s_i}{r}$. It enables to get r by computing some $\mathbf{lcm}(r_i', r_j')$.

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$\longrightarrow$ This procedure works with probability $(1-\mathrm{e}^{-C\log N})(1-\varepsilon)$ for constant $C>0$ depending on the number of repetitions

Final cost:

$$O\left(\log^3 N\right)$$

$\longrightarrow$ This could be done in time $O\Big(\log^2(N)\mathrm{poly}(\log\log N)\Big)$

Order finding algorithm is efficient because we know quantumly how to perform classical computations **and the quantum Fourier transform over $\mathbb{Z}/2^t\mathbb{Z}$**

SHOR'S ALGORITHM

Factoring problem:

- **Input:** an integer N
- **Output:** a non-trivial factor of N

$\longrightarrow$ Security of public-key encryption scheme RSA relies on the hardness of this problem. . .

Classically best algorithms have a complexity:

$$\exp\left((c + o(1)) \log^{\alpha}(N) \log^{1-\alpha}(\log N)\right)$$

Shor's algorithm is basically applying order finding for some random $x \in [\![0, N-1]\!] \ldots$

But why?

Theorem 1:

Suppose N is a L bits not prime integer and $1 \leq y \leq N$ be a non-trivial integer such that

$$y^2 = 1 \textbf{ mod } N$$

Then, at least $\mathbf{gcd}(y-1, N)$ or $\mathbf{gcd}(y+1, N)$ is a non-trivial factor of N that can be computed in time $O(L^3)$

Theorem 2:

Suppose that $N = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ where the p_i's are different primes. Let x be an integer chosen uniformly at random, subject to the requirements that $1 \leq x \leq N-1$ and $\mathbf{gcd}(x, N) = 1$. Let r be the order of x. Then,

$$\mathbb{P}\Big(r \text{ is even} \quad \text{and} \quad x^{r/2} \neq -1 \textbf{ mod } N\Big) \geq 1 - \frac{1}{2^m}$$

$\longrightarrow$ Let x be picked according to Theorem 2, then with $\Big(\text{at least}\Big)$ a constant probability

$x^{r/2}$ is a solution $\neq \pm 1$ of $\Big(X^2 = 1 \textbf{ mod } N\Big)$

According to Theorem 1: $\mathbf{gcd}(x^{r/2} - 1, N)$ or $\mathbf{gcd}(x^{r/2} + 1, N)$ is a $\neq \pm 1$ factor of N

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According to Theorem 1: $\textbf{gcd}(x^{r/2} - 1, N)$ or $\textbf{gcd}(x^{r/2} + 1, N)$ is a $\neq \pm 1$ factor of N

Given x, we just need to compute its order r to find a non-trivial factor!

1. Pick x uniformly at random in $[\![1, N]\!]$
2. Compute $d = \mathbf{gcd}(x, N)$. If $d > 1$, output d
3. Use the quantum order-finding subroutine to find the order r of x **mod** N
4. If r is even and $x^{r/2} \neq -1 \textbf{ mod } N$ then compute $\mathbf{gcd}(x^{r/2} - 1, N)$ or $\mathbf{gcd}(x^{r/2} + 1, N)$ and test if one of these is a non-trivial factor of N. Otherwise go back to Step 1.

By using the law of total probability:

$$
\begin{aligned}
\mathbb{P}\,(\text{success}) &\geq \mathbb{P}\,(\text{success} \mid \text{ Step 3 succeeds}) \cdot \mathbb{P}\,(\text{ Step 3 succeeds}) \\
&= \mathbb{P}\left(r \text{ is even and } x^{r/2} \neq -1 \textbf{ mod } N \right) \cdot \mathbb{P}\,(\text{ order finding succeeds}) \\
&\geq \left(1 - \frac{1}{2^m}\right) \cdot (1 - \mathrm{e}^{-C \log N})(1 - \varepsilon) \quad \left(m \text{ number of prime factors of } N\right)
\end{aligned}
$$

⟶ Repeating the algorithm a constant amount of times gives a non-trivial factor

Final cost:

$$O\left(\log^3 N\right) \text{ cost of phase estimation} + \text{Step 4}$$

HIDDEN SUBGROUP PROBLEM

Shor's algorithm relies on the order-finding which itself crucially used $\mathsf{QFT}_{\mathbb{Z}/2^t\mathbb{Z}}$

$\Big($in the phase estimation$\Big)$

$\longrightarrow$ It turns out that what we did is extremely "general"

Techniques we have presented enable to compute the "period" of a wide class of functions. . .

- What do we mean by "general"?
- Computing the "period" of which class of functions and does it imply some interesting statements?

$\longrightarrow$ Hidden Subgroup Problem!

Hidden Subgroup Problem (HSP):

- **Input:** a function $f : G \to S$ where G **is a known group**[a] and S is a finite set
- **Promise:** f satisfies

$$f(x) = f(y) \text{ if and only if } y \in xH$$
$$\text{i.e., } y = xh \text{ for some } h \in H$$

  for an **unknown subgroup** $H \subseteq G$
- **Output:** H

[a] see later for a precise definition

$\longrightarrow$ We say that f hides the subgroup H

Left-cosets:

The set:

$$xH \stackrel{\text{def}}{=} \{xh : h \in H\}$$

is called a **left-coset** of H

$\longrightarrow$ A function f that hides H is constant on each left-coset of H and distinct on different left cosets

HSP may be seen as a purely abstract problem. . . But no!

Here are particular instantiations of HSP

- Simon's problem:

$$G = \mathbb{F}_2^n,\ H = \left\{0, \mathbf{s}\right\} \text{ and } f \text{ being the input in Simon's problem}$$

- Order finding:

$$G = \mathbb{Z}/\Phi(N)\mathbb{Z}\ \Big(\Phi \text{ be the Euler function}\Big),\ H = \left\{rx : x \in \mathbb{Z}/\Phi(N)\mathbb{Z}\right\} \text{ and } f(a) = x^a \bmod N$$

- Discrete logarithm problem: see Exercise Session!
- etc. . .

Be careful:

In Shor's algorithm, when using a solver for the order finding problem we don't know $\Phi(N)$ and therefore we don't know G . . .

HOW TO SOLVE IT IN THE ABELIAN CASE: QFT! (KITAEV, INSPIRED BY SHOR)

We suppose that G is Abelian $\big($we note G in additive notation$\big)$

$f : G \longrightarrow S$ that hides some subgroup H

1. Start with $|0\rangle\,|0\rangle$, where the two registers have dimensions $\sharp G$ and $\sharp S$, respectively
2. Create a uniform superposition over G in the first register: $\frac{1}{\sqrt{\sharp G}} \sum_{g \in G} |g\rangle\,|0\rangle$
3. Compute f in superposition: $\frac{1}{\sqrt{\sharp G}} \sum_{g \in G} |g\rangle\,|f(g)\rangle$
4. Measure the second register. This yields some value $s \in G$. The second register collapses to $\big($using the promise over $f\big)$
$$\frac{1}{\sqrt{\sharp H}} \sum_{h \in H} |s + h\rangle$$
5. Apply QFT$_G$ giving: $\frac{1}{\sqrt{\sharp H}} \sum_{h \in H} |\chi_{s+h}\rangle$ for some quantum state $|\chi_{s+h}\rangle$ $\big(\chi_g$ characters of $G\big)$
6. Measure and output the resulting $g \in G$

G is Abelian, be $\left(\chi_g\right)_{g\in G}$ be its characters

$$\begin{aligned}
|\chi_{s+h}\rangle &= \mathbf{QFT}_G \sum_{h\in H} |s+h\rangle \\
&= \frac{1}{\sqrt{G}} \sum_{h\in H} \mathbf{QFT}_G\, |s+h\rangle \\
&= \frac{1}{\sqrt{G}} \sum_{h\in H} \sum_{g\in G} \chi_g(s+h)\, |g\rangle \\
&= \frac{1}{\sqrt{G}} \sum_{g\in G} \left(\sum_{h\in H} \chi_g(h) \right) \chi_g(s)\, |g\rangle \quad \text{from Lecture 5: } \sum_{h\in H} \chi_g(h) = \begin{cases} \sharp H & \text{if } g \in H^\perp \\ 0 & \text{otherwise} \end{cases} \\
&= \frac{1}{\sqrt{G}} \sum_{g\in H^\perp} \sharp H\ \chi_g(s)\, |g\rangle
\end{aligned}$$

$\longrightarrow$ The quantum step before measurement is: $\sqrt{\frac{\sharp H}{\sharp G}} \sum_{g\in H^\perp} \chi_g(s)\, |g\rangle$

The quantum state before measurement is:

$$\sqrt{\frac{\sharp H}{\sharp G}} \sum_{g \in H^{\perp}} \chi_g(s) \, |g\rangle \quad \text{where } H^{\perp} = \left\{ g \in G : \forall h \in H, \chi_g(h) = 1 \right\}$$

$\longrightarrow$ Measuring gives a uniform $g \in H^{\perp}$ giving some information about $H \ldots$

repeating a **poly**($\log \sharp G$) times enables to recover H with high probability!

- For a rigorous proof of this statement: see Chapter 6 in the lecture notes by Andrew Childs

An example: Simon's problem

$$G = \mathbb{F}_2^n, \quad \forall \mathsf{x}, \mathsf{y} \in \mathbb{F}_2^n, \; \chi_{\mathsf{x}}(\mathsf{y}) = (-1)^{\mathsf{x} \cdot \mathsf{y}} \quad \text{and } H = \left\{ \mathbf{0}, \mathsf{s} \right\}$$

$$\longrightarrow H^{\perp} = \left\{ \mathsf{x} \in \mathbb{F}_2^n : \mathsf{x} \cdot \mathsf{s} = 0 \right\}$$

In other words, we recover Simon's algorithm. . .

G is an Abelian group

Recall that we compute $\mathbf{QFT}_G$ as $\mathbf{QFT}_{\mathbb{Z}/n_1\mathbb{Z}} \otimes \cdots \otimes \mathbf{QFT}_{\mathbb{Z}/n_k\mathbb{Z}}$ where we used the isomorphism:

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z} \tag{1}$$

But is it easy to compute this isomorphism/decomposition even if we “know” G?

G is an Abelian group

Recall that we compute $\mathbf{QFT}_G$ as $\mathbf{QFT}_{\mathbb{Z}/n_1\mathbb{Z}} \otimes \cdots \otimes \mathbf{QFT}_{\mathbb{Z}/n_k\mathbb{Z}}$ where we used the isomorphism:

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z} \tag{1}$$

But is it easy to compute this isomorphism/decomposition even if we "know" G?

$\longrightarrow$ Yes! At least quantumly for a "good" definition of knowing G . . .

Quantum decomposition of Abelian groups:

Suppose we have (*i*) a unique encoding of each element of G, (*ii*) the ability to perform group efficiently operations on these elements, and (*iii*) a generating set for G.

Then, there exists an efficient quantum algorithm that decomposes G, namely outputs the isomorphism given in Equation (1)

$\longrightarrow$ See Chapter 6 in the lecture notes by Andrew Childs

GENERALIZATION TO THE NON-ABELIAN CASE?

To solve HSP we crucially used that we restrict ourself to the Abelian case...

$\Big($in the Abelian case, $H^\perp$ gives linear relations enabling to recover $H\Big)$

$\longrightarrow$ And the non-Abelian case?

No efficient algorithm is known for the non-Abelian case

$\Big($even if nothing indicates that it is impossible$\Big)$ ...

$\longrightarrow$ Finding such an algorithm would have a huge impact in theoretical computer science, $\Big($post-quantum$\Big)$ cryptography...

If you are interested by this topic:

- Nice reading about Fourier transform (classical & quantum) over non-Abelian group: Chapter 11 in the lectures by Andrew Child https://www.cs.umd.edu/~amchilds/qa/
- The hidden nonabelian subgroup problem and the Kuperberg algorithm, see Chapters 11-13 in the lectures by Andrew Child https://www.cs.umd.edu/~amchilds/qa/

EXERCISE SESSION