LECTURE 4
INTRODUCTION TO QUANTUM COMPUTING, THE CIRCUIT MODEL

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Computer science: art of computing...

What do we mean by quantum computing?

$\longrightarrow$ The quantum circuit model!

What is the cost to compute 2^n?

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- Trivial approach: compute $2 \times 2 \times 2 \times \ldots\ n$ times...

  $\longrightarrow$ It costs n operations!

- Clever approach: recursive algorithm, given n if $n > 1$ compute res $\leftarrow 2^{n/2}$ and compute res^2 otherwise output 2

  $\longrightarrow$ It costs $\approx \log_2(n)$ operations $\Big($exponential improvement$\Big)$!

Two lessons to take-away:

1. You have to be smart when computing something $\Big($algorithmic science$\Big)$
2. A first model of cost: enumerate the number of basic operations $\Big($additions and multiplications$\Big)$

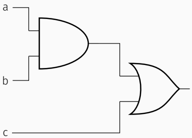
   $\longrightarrow$ It is an high level point of view, often convenient but rather "limited"

Boolean Circuits:

In what follows: focus on a "low" level to estimate the computational cost

$\longrightarrow$ boolean circuits & number of gates

Boolean circuit: finite directed acyclic $\Big($no loop$\Big)$ graph with **AND**, **OR** and **NOT** classical gates which has input and output nodes



A circuit computes $f : \{0,1\}^n \longrightarrow \{0,1\}^m$ if given n input bits **x**, it outputs m bits given by f(**x**)

Two questions:

- What are the classical gates that enable to compute any function $f : \{0,1\}^n \longrightarrow \{0,1\}^m$?
- How to define the efficiency of a circuit?

Universality:

Logic gates **AND**, **OR** and **NOT** are enough to compute any function $f : \{0,1\}^n \longrightarrow \{0,1\}^m$

$\left(\text{yes these gates enable to compute } n \mapsto 2^{n}\right)$

Is it doable quantumly?

Problem: *any quantum operation is invertible* $\left(\textit{even unitary}\right)$ but **AND** is not invertible. . .

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$\Big($yes these gates enable to compute $n \mapsto 2^n$ $\Big)$

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Problem: *any quantum operation is invertible* $\Big($*even unitary*$\Big)$ but **AND** is not invertible. . .

Toffoli $\Big($also CCNOT$\Big)$ gate:

The Toffoli gate takes 3 input bits and it outputs 3 bits as follows:

$$\textbf{Toffoli}(x, y, z) = \Big(x, y, z \textbf{ XOR } (x \textbf{ AND } y)\Big)$$

Inversability and universality:

- The Toffoli gate is invertible
- Any classical circuit computing a function f consisting of N gates in the set $\{\textbf{AND}, \textbf{OR}, \textbf{NOT}\}$ can be computed using $O(N)$ Toffoli gates

$\longrightarrow$ In particular: the number of Toffoli gates is roughly the same

Many different circuits can compute a function $f : \{0,1\}^n \longrightarrow \{0,1\}^m$

How can we distinguish them?

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How can we distinguish them?

$\longrightarrow$ Some circuits are more efficient than others!

Running time:

We define the running time of a circuit computing f as the number of used gates **AND**, **OR** and **NOT**

Ideal situation: an efficient circuit

Given n input nodes: the circuit uses $O(n^k)$ gates for some constant k

$\longrightarrow$ We say that it has a cost $\text{poly}(n)$

In this course: we only care of being $\text{poly}(n)$ $\Big($even if the constant k is large...$\Big)$

Exercise:

Is it equivalent to define our running-time model as the number of **Toffoli** gates to compute a function f? Why?

But is the classical circuit model meaningful?

P: class of languages $L \subseteq \{0,1\}^\star$ “for which it exists an efficient algorithm” to decide $x \in L$ or not

Complexity theory: uniformly polynomial circuits

Family of circuits $C \stackrel{\text{def}}{=} \{C_n\}_n$ with n input bits and one output bit *such that* there is polylog(n)-space Turing machine that outputs C_n given n

$$L_C \stackrel{\text{def}}{=} \bigcup_n \left\{ \mathbf{x} \in \{0,1\}^n \ : \ C_n(\mathbf{x}) = 1 \right\}$$

$L \in \mathrm{P}$ if and only if there exits a uniform family of circuits C such that $L = L_C$

$\longrightarrow$ Given a uniform family of circuits $C = \{C_n\}$: **C_n has at most poly(n)-gates!**

What about quantum computation?

Is the circuit model reasonable? If yes, what is doable quantumly and at which cost?

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Two intuitions:

- "Quantum circuit" can simulate classical circuits because Toffoli gates are universal and invertible. . .

  $\longrightarrow$ Therefore: quantum circuits define a "reasonable" model of computation

- Complexity of computation will be taken into account from the number of "quantum gates"

  $\longrightarrow$ Therefore: we expect quantum circuits to measure the complexity in a similar vein than in the classical case

NOTATION AND BASIC CIRCUITS

During this course we consider the state space $\mathbb{C}^{2^n} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$ of n-qubits register

State space, computational basis and measurement:

We will always write n-qubits registers as

$$\sum_{\mathbf{x} \in \{0,1\}^n} \alpha_{\mathbf{x}} |\mathbf{x}\rangle \quad \text{where } |\mathbf{x}\rangle = |x_1, \ldots, x_n\rangle \ \Big(= |x_1\rangle \otimes \cdots \otimes |x_n\rangle \Big) \text{ and } \sum_{\mathbf{x} \in \{0,1\}^n} |\alpha_{\mathbf{x}}|^2 = 1$$

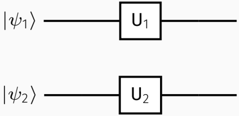
The family $(|\mathbf{x}\rangle)_{\mathbf{x} \in \{0,1\}^n}$ is known as the computational basis

$\longrightarrow$ All the considered measurements $\Big($in this course$\Big)$ will be in the computational basis

Given two quantum states $|\psi_1\rangle, |\psi_2\rangle$ and two unitaries $\mathsf{U}_1, \mathsf{U}_2$, the circuit representation of

$$(\mathsf{U}_1 \otimes \mathsf{U}_2)\,(|\psi_1\rangle \otimes |\psi_2\rangle)$$

is given by



Exercise:

1. What becomes $\frac{|00\rangle+|01\rangle}{\sqrt{2}}$ when feeding to the above circuit?
2. Describe a quantum circuit that transforms $|00\rangle$ into $\frac{|10\rangle-|11\rangle}{\sqrt{2}}$

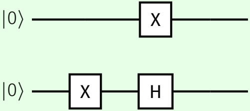
QUANTUM CIRCUIT AND TENSORIAL PRODUCT

Solution:

1. What becomes $\frac{|00\rangle+|01\rangle}{\sqrt{2}}$ when feeding to the above circuit?

   It becomes: $\mathbf{U}_1\,|0\rangle \otimes \mathbf{U}_2\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\mathbf{U}_1\,|0\rangle \otimes \mathbf{U}_2\,|0\rangle + \frac{1}{\sqrt{2}}\mathbf{U}_1\,|0\rangle \otimes \mathbf{U}_2\,|1\rangle$

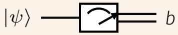
2. Describe a quantum circuit that transforms $|00\rangle$ into $\frac{|10\rangle-|11\rangle}{\sqrt{2}}$



A measurement *in the computational basis* converts $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ into a probabilistic classical bit $b \in \{0, 1\}$ where

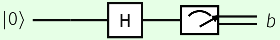
$$\mathbb{P}(b = 0) = |\alpha|^2 \quad \text{and} \quad \mathbb{P}(b = 1) = |\beta|^2$$

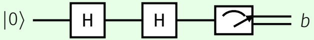
The circuit representation of a measurement is:



Exercise:

Give the distribution of the following probabilistic bits b:

1. 

2. 

Solution:

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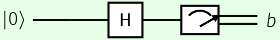
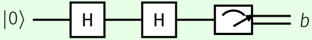
1. 

   The output bit b is uniform, namely: $\mathbb{P}(b = 0) = \mathbb{P}(b = 1) = \frac{1}{2}$

2. 

   As $\mathbf{H}^2 = \mathbf{I}_2$, the output bit b is always zero

THE QUANTUM CNOT GATE:

Let us introduce the **Controlled-NOT** gate $\Big($unitary$\Big)$ over 2-qubits:

$$\mathsf{CNOT} : |a, b\rangle \mapsto |a, a \oplus b\rangle$$

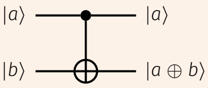
It is a unitary $\Big($it maps the computational basis to the computational basis$\Big)$

Quantum CNOT-gate $|a, b\rangle \mapsto |a, a \oplus b\rangle$

- **Matrix representation:**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- **Circuit representation:**



$$|a, b\rangle \mapsto |a, a \oplus b\rangle$$

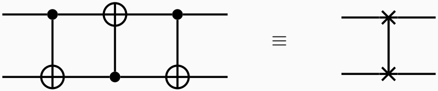
is the quantum generalization of the **XOR** operation!

Be careful:

The **XOR** operation $(a, b) \mapsto a \oplus b$ cannot be a quantum operation because is not invertible

SWAPPING

Given two wires, is it possible to swap two qubits?



$$\begin{aligned} |a, b\rangle &\longrightarrow |a, a \oplus b\rangle \\ &\longrightarrow |a \oplus (a \oplus b)\,, a \oplus b\rangle \\ &\longrightarrow |b, (a \oplus b) \oplus b\rangle \\ &= |b, a\rangle \end{aligned}$$

Given a qubit $|\psi\rangle$, is it possible to build a quantum circuit that copies it?

$\longrightarrow$ **No!** Because the no-cloning theorem

But it is doable for classical bit $(b, 0) \mapsto (b, 0 \oplus b) = (b, b) \ldots$

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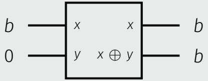
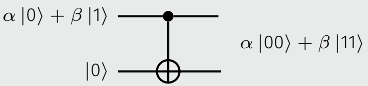
Take a look at the quantum case:



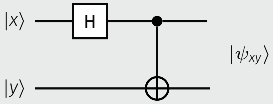


We have built an entangled state!

Bell states:

$$|\psi_{xy}\rangle \stackrel{\text{def}}{=} \frac{|0, y\rangle + (-1)^x \, |1, (1 \oplus y)\rangle}{\sqrt{2}}$$

The quantum circuit building Bell states:



$$|x, y\rangle \xrightarrow{\text{H}\otimes\text{I}_2} \frac{|0\rangle + (-1)^x \, |1\rangle}{\sqrt{2}} \otimes |y\rangle = \frac{|0, y\rangle + (-1)^x \, |1, y\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|0, y\rangle + (-1)^x \, |1, (1 \oplus y)\rangle}{\sqrt{2}}$$

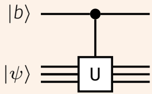
CONTROLLED UNITARY

Controlled U-gate:

Let $\mathbf{U}$ be any unitary over n-qubits. The controlled $\mathbf{U}$-gate has one control qubit $|b\rangle$ and n target qubits $|\psi\rangle$. It is defined as

- If $b = 0$, it outputs $|b\rangle \otimes |\psi\rangle$
- If $b = 1$, it outputs $|b\rangle \otimes \mathbf{U}\,|\psi\rangle$

Circuit representation:



$\longrightarrow$ Controlled-$\mathbf{U}$ $\equiv$ If condition then instruction $\mathbf{U}$ otherwise do nothing

Exercise:

Show that the CNOT gate is the controlled $\mathbf{X}$-gate

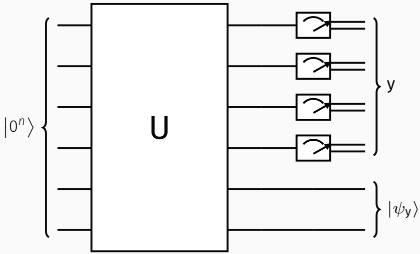
Quantum circuits: starting from n qubits initialized at $|0^n\rangle$ and then successively apply the two admissible operations $\big($unitary and measurements$\big)$

Applying $\mathbf{U}_1$ and then $\mathbf{U}_2$ is equivalent to applying $\mathbf{U}_2\mathbf{U}_1$

$\longrightarrow$ We can assume the algorithm performs a unitary, then a measurement, then a unitary, then measurement and so on...

We will consider only algorithms where we first perform all the unitary operations and then perform measurements in the computational basis

$\longrightarrow$ As powerful as general algorithms $\big($admitted$\big)$



AUXILIARY QUBITS

$$\mathsf{U} : |\psi\rangle \longrightarrow \mathsf{U}\,|\psi\rangle$$

$\longrightarrow$ It is often easier to build $\mathsf{U}' : |\psi\rangle\,|0\rangle_{\text{aux}} \longrightarrow \mathsf{U}\Big(\,|\psi\rangle\,\Big)\,|0\rangle_{\text{aux}}$

Extra qubits are called auxiliary qubits, ancilliary qubits or workspace

$\longrightarrow$ it is important that they start at $|0\rangle$ and end at $|0\rangle$ $\Big($see Exercise Session$\Big)$

SOLOVAY-KITAEV THEOREM AND GATE MODEL

Any classical function can be computed with gates $\{\textbf{AND}, \textbf{OR}, \textbf{NOT}\}$ (universal gates)

What are the universal quantum gates?

The following gate is crucial:

The $\pi/8$-gate:

It maps $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto e^{i\pi/4}|1\rangle$:

$$\mathbf{T} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Origin of the terminology:

Up to an unimportant global phase $\mathbf{T}$ is equal to $\mathbf{T} = e^{i\pi/8}\begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$

$\left\{\mathsf{CNOT}, \mathsf{H}, \mathsf{T}\right\}$ *are universal quantum gates*

Solovay-Kitaev theorem (admitted):

Let $\mathcal{G} = \{\mathsf{CNOT}, \mathsf{H}, \mathsf{T}\}$. Any unitary $\mathbf{U}$ over n-qubits can be approximated by applying

$$O\left(2^{2n} \log^4\left(\frac{1}{\varepsilon}\right)\right)$$

gates from $\mathcal{G}$ with accuracy ε

In other words, from the description of $\mathbf{U}$, one can construct a sequence $\mathbf{G}_1, \ldots, \mathbf{G}_N \in \mathcal{G}$ with $N = O(2^{2n} \log^4(\frac{1}{\varepsilon}))$ and

$$\|\mathbf{G}_N \ldots \mathbf{G}_1 - \mathbf{U}\| \leq \varepsilon,$$

where $\|\mathbf{G}_N \cdots \mathbf{G}_1 - \mathbf{U}\| \stackrel{\text{def}}{=} \max_{|\psi\rangle} \|\mathbf{G}_N \cdots \mathbf{G}_1 |\psi\rangle - \mathbf{U} |\psi\rangle\|$ is the *operator norm*

$\longrightarrow$ The log term is important: exponential accuracy with a polynomial number of gates

Other universal quantum gates?

Yes! The **CNOT** and qubits gates are also universal

How many resources are needed to compute a fixed unitary **U over n qubits**?

- **First definition:** it requires one resource, the unitary **U**

$\longrightarrow$ *Stupid definition*: same thing that saying, to compute classically any function f asks one resource, the function f

We want a the smallest and simplest set of operations to define the needed resources

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No! As for classical computations there are algorithms/unitaries easy to compute, other not. . .

A reasonable model to define the cost of a quantum computation, i.e. computing a unitary

. The number of $\left\{\text{CNOT}, \text{H}, \text{T}\right\}$ to approximate well-enough the unitary

But would you be happy to implement X *or* Y *with this set of quantum gates?*

$\longrightarrow$ *A priori* ***no***! The set of operations $\left\{\text{CNOT}, \text{H}, \text{T}\right\}$ is not very flexible...

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Wouldn't be more reasonable to use as model of cost: the number of unitaries over 1 and 2-qubits?

Yes and by Solovay-Kitaev both models are "poly(λ)-equivalent"

We can approximate any unitary over 1 and 2 qubits with accuracy $2^{-\lambda}$ and

$O\left(\lambda^4\right)$ quantum gates $\left\{\mathsf{CNOT}, \mathsf{H}, \mathsf{T}\right\}$

The quantum gate model:

The quantum running time of a unitary $\mathbf{U}$ is the amount of 1 and 2-qubit gates needed to apply $\mathbf{U}$

The running time of a single-qubit measurement is 1

Exercise:

Give a simple argument to explain why quantum gates over 1-qubit are not universal, *i.e.* are not enough to describe any quantum computation

One may say that estimating the running time as the number of 1-2 qubits unitaries is an overkill

$\longrightarrow$ It can be hard to build some 1 or 2 qubits unitary...

A more reasonable model:

Running time: number **H**, **T** and **CNOT** gates that are used

$\longrightarrow$ The "difficulty" to implement quantum circuits reduces to build this small set of gates!

By the Solovay-Kitaev theorem:

The running time of the above model is the same than in the quantum gate model, but up to polynomial factor $\left(\text{in the input length } n\right)$ if one targets an exponentially close accuracy...

In conclusion: lot of debates to define the running time of quantum circuits...

For us: no debates, we don't care of polynomial factors (even if it is a hard problem to handle in "practice"...) and we will use the quantum gate model

To take away: you said algorithm?

- Algorithm: series of simple and determined in advance instructions $\Big($*addition, multiplication, if condition then instruction, while condition do instruction*$\Big)$

  $\longrightarrow$ Efficient algorithm: small number of instructions!

- Quantum algorithm: series of 1, 2-qubits unitaries and then measurements

  $\longrightarrow$ Efficient quantum algorithm: small amount of 1, 2-qubits unitaries and measurements!

Efficient quantum algorithm: poly(n)-repetitions of a circuit starting from $|0^n\rangle$ with poly(n) unitaries and measurements over 1, 2-qubits

Efficient computing: a difficult task

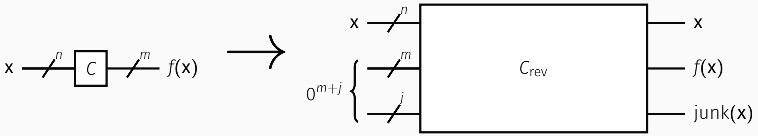
For many problems, it is $\Big($very$\Big)$ hard to find a small number of instructions solving it

Shor's quantum algorithm has been a breakthrough: it solves with "few" quantum-instructions a problem $\Big($factoring$\Big)$ such that all known classical algorithms ask a huge number of instructions...

CLASSICAL CIRCUITS WITH QUANTUM CIRCUITS

CLASSICAL CASE

Computing classically a function f with T gates can be transformed into a reversible circuit C_{rev} that only consists of $O(T)$ Toffoli gates, possibly with some junk state $\mathrm{junk}(\mathbf{x})$.



Informally, the junk part keeps a place to perform intermediary computations

Simulating classical circuits with quantum circuits:

Classical Toffoli gates can be interpreted as a quantum unitary acting on three qubits:

$$\textbf{Toffoli}\,|x, y, z\rangle \overset{\text{def}}{=} |x, y, z \oplus xy\rangle$$

Therefore: C_{rev} can be interpreted as a unitary $\mathbf{U}$:

$$\mathbf{U}\left|\mathbf{x}, 0^{m+j}\right\rangle \overset{\text{def}}{=} |\mathbf{x}\rangle\,|f(\mathbf{x})\rangle\,|\mathrm{junk}(\mathbf{x})\rangle$$

$\longrightarrow$ Quantum computers are at least as powerful as classical computers!

The unitary U_f:

For any function $f : \{0,1\}^n \to \{0,1\}^m$ that can be computed classically with a circuit running in time T, there exists a quantum circuit on $n+m$ qubits that runs in time $O(T)$ that can perform the unitary

$$U_f : |x\rangle\,|y\rangle \to |x\rangle\,|y \oplus f(x)\rangle$$

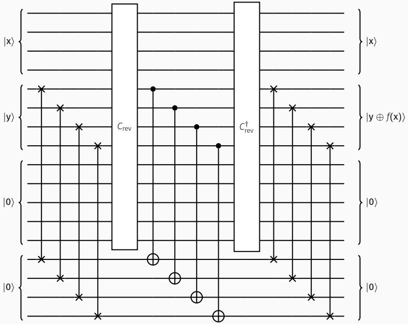
Be careful:

$|x\rangle \mapsto |f(x)\rangle$ may not be a quantum operation $\Big($for instance f be the zero function$\Big)$

$\longrightarrow$ The auxiliary qubit $|y\rangle$ ensures that **U_f is a unitary!**

Proof:

1. On input $|x\rangle\,|y\rangle\,|0\rangle\,|0\rangle$, first swap the second and fourth registers to get $|x\rangle\,|0\rangle\,|0\rangle\,|y\rangle$.
2. Apply C_{rev} on the 3 first registers to get the state $|x\rangle\,|f(x)\rangle\,|\mathrm{junk}(x)\rangle\,|y\rangle$.
3. For i from 1 to m, apply a **CNOT** gate between the i^{th} wire of the second register and the i^{th} wire of the forth register. We then have the state $|x\rangle\,|f(x)\rangle\,|\mathrm{junk}(x)\rangle\,|y \oplus f(x)\rangle$.
4. Apply $C_{\mathrm{rev}}^{\dagger}$ on the three first registers to get the state $|x\rangle\,|0\rangle\,|0\rangle\,|y \oplus f(x)\rangle$.
5. Swap the second and forth register to get the state $|x\rangle\,|y \oplus f(x)\rangle\,|0\rangle\,|0\rangle$.

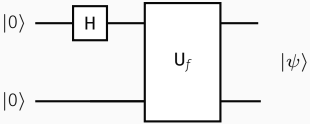


QUANTUM PARALLELISM AND INTERFERENCE

$$\text{Let } f : \{0,1\} \to \{0,1\}$$

$$U_f : |x\rangle |y\rangle \to |x\rangle |y \oplus f(x)\rangle$$

Consider the following quantum circuit:



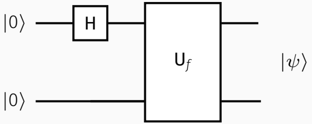
What quantum state is $|\psi\rangle$?

QUANTUM PARALLELISM: ONE BIT FUNCTIONS

$$\text{Let } f : \{0,1\} \to \{0,1\}$$

$$\mathbf{U}_f : |x\rangle\, |y\rangle \to |x\rangle\, |y \oplus f(x)\rangle$$

Consider the following quantum circuit:



What quantum state is $|\psi\rangle$?

1. After the first gate we have: $\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{|00\rangle+|10\rangle}{\sqrt{2}}$,
2. Applying $\mathbf{U}_f$ leads to $\Big($use the linearity$\Big)$:

$$|\psi\rangle = \frac{|0,f(0)\rangle+|1,f(1)\rangle}{\sqrt{2}}$$

$\longrightarrow$ We have a superposition of the values of f

Tensorization of the Hadamard gate:

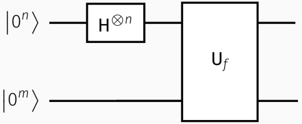
Consider,

$$\mathsf{H}^{\otimes n} \stackrel{\text{def}}{=} \underbrace{\mathsf{H} \otimes \cdots \otimes \mathsf{H}}_{n \text{ times}}$$

Then,

$$\mathsf{H}^{\otimes n} \left|0^n\right\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

The following circuit performs the quantum parallelism $\left(\text{here } f : \{0,1\}^n \to \{0,1\}^m\right)$



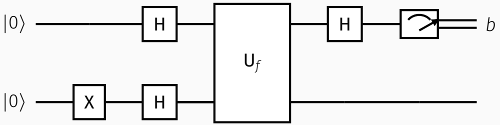
$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x, f(x)\rangle$$

Measurement of $\frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}} |\mathbf{x}, f(\mathbf{x})\rangle$ gives $f(\mathbf{x})$ for only one value of $\mathbf{x}$ . . .

$\longrightarrow$ Interference is a nice example of how using quantum parallelism!

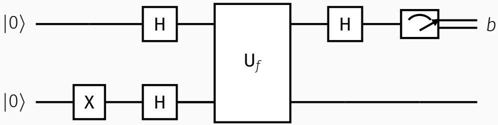
The "−1" of the Hadamard gate gives you a huge power. . .

Consider the following circuit $\left(\text{here } f : \{0,1\} \to \{0,1\}\right)$



What is the value of b?

Consider the following circuit $\left(\text{here } f : \{0,1\} \to \{0,1\}\right)$



What is the value of b?

1. After applying the **X** and **H** gates: $\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$,

2. Applying $\mathbf{U}_f$ leads to (use the linearity):

$$\frac{|0, f(0)\rangle - |0, 1\oplus f(0)\rangle + |1, f(1)\rangle - |1, 1\oplus f(1)\rangle}{2} = \begin{cases} \pm \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) = f(1) \\ \pm \frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) \neq f(1) \end{cases}$$

3. Applying the last Hadamard gate leads to (use that $\mathbf{H}^2 = \mathrm{I}_2$):

$$\begin{cases} \pm |0\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) = f(1) \\ \pm |1\rangle \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) \neq f(1) \end{cases}$$

4. Measuring the first qubit always leads to $f(0) \oplus f(1)$!

$\longrightarrow$ We obtained a global property of f $\left(\textit{i.e.}, f(0) \oplus f(1)\right)$ with only one evaluation of $f(x)$!

SIMON'S ALGORITHM

The problem:

- **Input:** A function $f : \{0,1\}^n \longrightarrow \{0,1\}^n$
- **Promise:** $\exists \mathsf{s} \in \{0,1\}^n$: $\Big(f(\mathsf{x}) = f(\mathsf{y}) \iff (\mathsf{x} = \mathsf{y})$ or $(\mathsf{x} = \mathsf{y} \oplus \mathsf{s})\Big)$
- **Goal:** Find s

1. Start from the $2n$ qubit state, with 2 registers of n qubits

$$|\psi_0\rangle = |0^n\rangle |0^n\rangle$$

2. Apply $\mathsf{H}^{\otimes n}$ on the first n qubits to get

$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |0^n\rangle$$

3. Apply U_f on the state to get

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle = \frac{1}{\sqrt{\sharp \mathrm{Im}(f)}} \sum_{y \in \mathrm{Im}(f)} \frac{1}{\sqrt{2}} \left(|x_y\rangle + |x_y \oplus s\rangle\right) |y\rangle$$

4. Measure the second register and obtain some value $y \in \mathrm{Im}(f)$. The resulting state on the first register is

$$|\psi_4(y)\rangle = \frac{1}{\sqrt{2}} \left(|x_y\rangle + |x_y \oplus s\rangle\right)$$

5. Apply $\mathsf{H}^{\otimes n}$ on the first register to get

$$|\psi_5(y)\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} \left(\frac{1}{\sqrt{2}} (-1)^{x_y \cdot z} + \frac{1}{\sqrt{2}} (-1)^{(x_y \oplus s) \cdot z} \right) |z\rangle$$

5. Apply $\mathsf{H}^{\otimes n}$ on the first register to get

$$|\psi_5(\mathsf{y})\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathsf{z}\in\{0,1\}^n} \left(\frac{1}{\sqrt{2}}(-1)^{\mathsf{x_y}\cdot\mathsf{z}} + \frac{1}{\sqrt{2}}(-1)^{(\mathsf{x_y}\oplus\mathsf{s})\cdot\mathsf{z}} \right) |\mathsf{z}\rangle \,.$$

Now, if $\mathsf{s}\cdot\mathsf{z} = 0 \bmod 2$, we have $\left(\frac{1}{\sqrt{2}}(-1)^{\mathsf{x_y}\cdot\mathsf{z}} + \frac{1}{\sqrt{2}}(-1)^{(\mathsf{x_y}\oplus\mathsf{s})\cdot\mathsf{z}} \right) = \sqrt{2}(-1)^{\mathsf{x_y}\cdot\mathsf{z}}$ and if $\mathsf{s}\cdot\mathsf{z} = 1 \bmod 2$, we have $\left(\frac{1}{\sqrt{2}}(-1)^{\mathsf{x_y}\cdot\mathsf{z}} + \frac{1}{\sqrt{2}}(-1)^{(\mathsf{x_y}\oplus\mathsf{s})\cdot\mathsf{z}} \right) = 0$. Therefore, we can write

$$|\psi_5(\mathsf{y})\rangle = \sqrt{\frac{2}{2^n}} \sum_{\substack{\mathsf{z}\in\{0,1\}^n \\ \mathsf{s}\cdot\mathsf{z}=0 \bmod 2}} (-1)^{\mathsf{x_y}\cdot\mathsf{z}} |\mathsf{z}\rangle \,.$$

6. Measure this state in the computational basis. You get a random z satisfying $\mathsf{z}\cdot\mathsf{s} = 0$

WHY DOES IT WORK?

$\mathbb{F}_2$ denotes $\{0,1\}$ modulo 2. It is a field

$\mathbb{F}_2^n$ is a n-dimensional $\mathbb{F}_2$ vector space

$\left\{\mathbf{z} \in \mathbb{F}_2^n : \ \mathbf{z} \cdot \mathbf{s} = \sum_{i=1}^n z_i s_i = 0 \in \mathbb{F}_2\right\}$ is a subspace of $\mathbb{F}_2^n$ with dimension $n-1$

The above algorithm gives $(z_1, \ldots, z_n)$ s.t $\sum_{i=1}^n z_i s_i = 0 \textbf{ mod } 2$

We repeat the algorithm m times to get m random values $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)} \in \mathbb{F}_2^n$ satisfying $\mathbf{z}^{(k)} \cdot \mathbf{s} = 0$

We obtain the following system $\Big(\mathbf{s}$ is the unknown$\Big)$: $\mathbf{Z}\mathbf{s} = \mathbf{0}$ where $\mathbf{Z} \stackrel{\text{def}}{=} \left(z_j^{(i)}\right)_{\substack{1 \le i \le n \\ 1 \le j \le m}}$

$\longrightarrow$ If $\mathbf{Z} \in \mathbb{F}_2^{m \times n}$ has rank $n-1$, we perform a Gaussian elimination to recover $\mathbf{s}$!

It will be verified with high probability if m large enough, $m = Cn$ for some constant $C > 0$

CONCLUSION: RUNNING TIME OF SIMON'S ALGORITHM

T be the classical running-time of f

Running time in the quantum gate model of one iteration:

- In Step 3 we apply $\mathbf{U}_f$: it can be done by using $O(T)$ quantum gates over qubits
- In Steps 2 and 5 we apply $2n$ times $\mathbf{H}$
- In Step 4 we perform a measurement on n-registers qubits: n measurements over qubits $\Big($in the computational basis$\Big)$

One iteration:

It costs quantumly $4n + O(T)$

- We repeat $O(n)$ times an iteration: it costs $O(n^2 + nT)$
- We solve a system by a classical Gaussian elimination: it costs $O(n^3)$

Overall cost in the quantum gate model:

Simon's algorithm costs $O(n^2 + n^3 + nT) = O(n^3 + nT)$

A LAST CONCLUSION

- We have solved Simon's problem in polynomial time with high probability with **only $O(n)$ queries to f** $\left(\textit{i.e.}, O(n) \text{ calls to } \mathbf{U}_f, \text{ step } 3\right)$

Is it doable classically?

- Simon has proved that any classical randomized algorithm that finds $\mathbf{s}$ with high probability needs **to make $\geq C\sqrt{2^n}$ queries to f** where C constant

$\longrightarrow$ **Quantum computing provides an exponential advantage!**

There are many results about the query complexity of quantum algorithms

- *Ronald de Wolf's lecture notes, Chapters* 11-12.

`https://arxiv.org/pdf/1907.09415.pdf`

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But one may say that solving Simon's problem is useless. . .

Simon's algorithm has been "the starting point" of Shor's algorithm that quantumly breaks all current deployed public-key cryptography

$\longrightarrow$ Come at Lecture 6!

EXERCISE SESSION