LECTURE 3
DENSITY OPERATOR AND PARTIAL TRACE

Quantum computer science and applications

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To answer the following questions:

- How can we model the quantum state after a measurement?

  ex: $|0\rangle$ with prob. 1/2 and $|1\rangle$ with prob. 1/2

- How can we describe the quantum state relative to a subsystem?

  ex: the first qubit of the EPR-pair $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$

$\longrightarrow$ Density operator/matrix and partial trace!

COURSE OUTLINE

1. General Properties of Density Operators
2. The Reduced Density Operator, Partial Trace and Application to The teleportation
3. Schmidt Decomposition and Purification

$\longrightarrow$ This course gives the basis of quantum information theory!

DENSITY OPERATOR

OBSERVABLE: A REMINDER

Observable: an equivalent description of projective measurements

- Observable: M an Hermitian operator $\left(\textit{i.e.}\ \mathsf{M}^\dagger = \mathsf{M}\right)$
- M is diagonalizable in an orthonormal basis: orthogonal projectors P_m onto the eigenspaces define the measurement
- Given $|\psi\rangle$, average outcome value:

$$\langle \mathsf{M} \rangle = \langle \psi | \mathsf{M} | \psi \rangle = \mathbf{tr}\left(\mathsf{M} |\psi\rangle\langle\psi| \right)$$

An example:

X defines a measurement with outcome ± 1:

$$\mathsf{X} = |+\rangle\langle+| + (-1)\,|-\rangle\langle-|$$

Given $|0\rangle$ $\left(\textit{resp.}\ |1\rangle\right)$, the average outcome value is 0:

$$\langle 0 | \mathsf{X} | 0 \rangle = \langle 0 | + \rangle \langle + | 0 \rangle - \langle 0 | - \rangle \langle - | 0 \rangle = \frac{1}{2} - \frac{1}{2} = 0$$

$$\langle 1 | \mathsf{X} | 1 \rangle = \langle 1 | + \rangle \langle + | 1 \rangle - \langle 1 | - \rangle \langle - | 1 \rangle = \frac{1}{2} - \frac{1}{2} = 0$$

Suppose that ρ is a probabilistic mixture of quantum states:

ρ : $|\psi_j\rangle$ **with probability** p_j

What is the average outcome value given ρ and an observable **M**?

By law of total probabilities,

$$\sum_j p_j \langle\psi_j| \mathbf{M} |\psi_j\rangle = \sum_j p_j \, \mathbf{tr}\left(\mathbf{M} |\psi_j\rangle\langle\psi_j|\right)$$
$$= \mathbf{tr}\left(\mathbf{M} \sum_j p_j |\psi_j\rangle\langle\psi_j|\right)$$

It justifies to introduce probabilistic mixture of quantum states as:

Define the probabilistic mixture ρ as:

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

The density matrix:

The density matrix ρ corresponding to a probabilistic mixture of states $(|\psi_j\rangle)_j$, the corresponding quantum state being equal to $|\psi_j\rangle$ with probability p_j, is given by

$$\rho \stackrel{\text{def}}{=} \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

$\longrightarrow \left\{p_i, |\psi_i\rangle\right\}$ is a set of states generating a density matrix ρ

The density matrix of a qubit:

$$|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \quad \langle\psi| = \begin{pmatrix}\overline{\alpha} & \overline{\beta}\end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \begin{pmatrix}|\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2\end{pmatrix}$$

- Density matrix of $|0\rangle$ $\left(\textit{resp.}\ |1\rangle\right)$ is

$$\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix} \quad \left(\text{resp.}\ \begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\right)$$

- Density matrix of $|+\rangle$ $\left(\textit{resp.}\ |-\rangle\right)$ is

$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \quad \left(\text{resp.}\ \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\right)$$

Exercise:

- Compute the density matrix of:
  1. the probabilistic mixture of $|0\rangle$ with prob. $\frac{1}{2}$ and $|1\rangle$ with probability $\frac{1}{2}$
  2. the probabilistic mixture of $|+\rangle$ with prob. $\frac{1}{2}$ and $|-\rangle$ with probability $\frac{1}{2}$
  3. what can you conclude?
- Compare the density matrix of $|\psi\rangle$ with $e^{i\theta}|\psi\rangle$. What can you conclude?

SOLUTION

- We have:
  1. Probabilistic mixture of $|0\rangle$ with prob. $\frac{1}{2}$ and $|1\rangle$ with prob. $\frac{1}{2}$:

  $$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

  2. Probabilistic mixture of $|+\rangle$ with prob. $\frac{1}{2}$ and $|-\rangle$ with prob. $\frac{1}{2}$:

  $$\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

  3. These probabilistic mixtures have the same density operator: they are indistinguishable

- $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ have the same density operator: they are indistinguishable

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- $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ have the same density operator: they are indistinguishable

$\longrightarrow$ Wait a little bit to be convinced that “they are indistinguishable”

We could (why not?) have stated quantum mechanics **using density operators as the primary model of states!**

In particular: postulates of quantum mechanics given with density matrix point of view

Let U be a unitary. Suppose that $|\psi\rangle$ is in the state $|\psi_i\rangle$ with probability p_i

$\longrightarrow$ After applying U: $|\psi\rangle$ will be in the state $U|\psi_i\rangle$ with probability p_i

$$\left(|\psi_i\rangle\langle\psi_i| \xrightarrow{U} U|\psi_i\rangle\langle\psi_i| U^\dagger\right)$$

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$$\left(|\psi_i\rangle\langle\psi_i| \xrightarrow{\mathsf{U}} \mathsf{U}|\psi_i\rangle\langle\psi_i|\mathsf{U}^\dagger\right)$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \xrightarrow{\mathsf{U}} \sum_i p_i \mathsf{U}|\psi_i\rangle\langle\psi_i|\mathsf{U}^\dagger = \mathsf{U}\rho\mathsf{U}^\dagger$$

Let $(\mathsf{M}_m)_m$ be a quantum measurement. Suppose that $|\psi\rangle$ is in the state $|\psi_i\rangle$ with probability p_i

- If the initial state is $|\psi_i\rangle$, the probability to measure m is:

$$p(m|i) = \langle\psi_i| \mathsf{M}_m^\dagger \mathsf{M}_m |\psi_i\rangle = \mathrm{tr}\left(\mathsf{M}_m^\dagger \mathsf{M}_m |\psi_i\rangle\langle\psi_i|\right)$$

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- Using the law of total probability, we measure m with probability:

$$p(m) = \sum_i p(m|i)p_i = \sum_i \mathbf{tr}\left(\mathbf{M}_m^\dagger \mathbf{M}_m |\psi_i\rangle\langle\psi_i|\right) p_i = \mathbf{tr}\left(\mathbf{M}_m^\dagger \mathbf{M}_m \rho\right)$$

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- If the initial state is $|\psi_i\rangle$ and we have measured m, the state becomes:

$$|\psi_i^m\rangle = \frac{\mathbf{M}_m |\psi_i\rangle}{\sqrt{\mathbf{tr}\left(\mathbf{M}_m^\dagger \mathbf{M}_m |\psi_i\rangle\langle\psi_i|\right)}} = \frac{\mathbf{M}_m |\psi_i\rangle}{\sqrt{p(m|i)}}$$

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The corresponding density operator ρ_m is:

$$\rho_m = \sum_i p(i|m) |\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(i|m) \frac{\mathbf{M}_m |\psi_i\rangle\langle\psi_i| \mathbf{M}_m^\dagger}{p(m|i)} = \sum_i \frac{p_i}{p(m)} \mathbf{M}_m |\psi_i\rangle\langle\psi_i| \mathbf{M}_m^\dagger$$

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$$\rho_m = \frac{\mathbf{M}_m \rho \mathbf{M}_m^\dagger}{\mathbf{tr}\left(\mathbf{M}_m^\dagger \mathbf{M}_m \rho\right)}$$

- Unitary Evolution U:

$$\rho \xrightarrow{\mathsf{U}} \mathsf{U}\rho\mathsf{U}^\dagger$$

- Measurement $(\mathsf{M}_m)_m$:

1. Probability to measure m:

$$\operatorname{tr}\left(\mathsf{M}_m^\dagger \mathsf{M}_m \rho\right)$$

2. After measuring m:

$$\frac{\mathsf{M}_m \rho \mathsf{M}_m^\dagger}{\operatorname{tr}\left(\mathsf{M}_m^\dagger \mathsf{M}_m \rho\right)}$$

Theorem:

An operator ρ acting on an Hilbert space is a density operator if and only if

1. ρ is positive
2. $\mathbf{tr}\left(\rho\right) = 1$

$\longrightarrow$ This characterization does not rely on a set of interpretation!

In particular: give a description of quantum mechanics with density operators that does not take as its foundation the state vector

Proof:

$\Rightarrow$: Suppose $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Then

$$\mathbf{tr}(\rho) = \sum_i p_i \, \mathbf{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i \, \mathbf{tr}(\langle\psi_i|\psi_i\rangle) = \sum_i p_i = 1$$

as $(p_i)_i$ defines a distribution. Let $|\psi\rangle$ be an arbitrary vector in the state space

$$\langle\psi| \rho |\psi\rangle = \sum_i p_i \langle\psi|\psi_i\rangle \langle\psi_i|\psi\rangle = \sum_i p_i |\langle\psi|\psi_i\rangle|^2 \geq 0$$

$\Leftarrow$: Suppose ρ positive operator with trace one

By the spectral decomposition theorem, there exists an orthonormal basis $(|i\rangle)_i$ $\Big($in particular the $|i\rangle$'s have norm 1$\Big)$ with associated positive eigenvalue $(\lambda_i)_i$ such that

$$\rho = \sum_i \lambda_i |i\rangle\langle i|$$

But,

$$\mathbf{tr}(\rho) = \sum_i \lambda_i = 1$$

Therefore ρ is a $(\lambda_i)_i$-probabilistic mixture of the quantum states $(|i\rangle)_i$

Pure state:

A state is called pure if it cannot be represented as a mixture (convex combination) of other states

This is equivalent to the density matrix being a one dimensional projector, *i.e.* $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ is a state $\big($a unit vector$\big)$

Mixed States:

A quantum system which is not in pure state is said to be in mixed states

Example:

1. $|0\rangle$, $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $|01\rangle$ and $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ are pure states
2. The probabilistic state "$|0\rangle$ with probability $\frac{1}{2}$ and $|1\rangle$ with probability $\frac{1}{2}$" is a mixed state

Theorem:

Any density operator ρ verifies

$$\mathbf{tr}\left(\rho^2\right) \leq 1$$

Furthermore,

$$\mathbf{tr}\left(\rho^2\right) = 1 \iff \rho \text{ is a pure state}$$

Proof:

First: any density operator ρ can be written as $\sum_i \lambda_i |i\rangle\langle i|$ where $(|i\rangle)_i$ orthonormal basis, $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$ $\Big($consequence of the fact that ρ positive operator and $\mathbf{tr}(\rho) = 1\Big)$. Therefore,

$$\rho^2 = \sum_i \lambda_i^2 |i\rangle\langle i|$$

Using that $(\lambda_i)_i$ is a distribution concludes the proof

It may be tempting to interpret: $\rho \stackrel{\text{def}}{=} \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ as "$|0\rangle$ with prob. $\frac{1}{2}$ and $|1\rangle$ with prob. $\frac{1}{2}$".
But, ρ also verifies: $\rho = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \dots$

$\longrightarrow$ They define the same statistics of measurements!

In quantum computing we deal with quantum states

Exercise:

1. Suppose that we are given "$|0\rangle$ with probability $\frac{1}{2}$ and $|1\rangle$ with probability $\frac{1}{2}$"
    - Measure in the basis $(|0\rangle, |1\rangle)$, what do you obtain as distribution of outcomes?
    - Measure in the basis $(|+\rangle, |-\rangle)$, what do you obtain as distribution of outcomes?
2. Suppose that we are given "$|+\rangle$ with probability $\frac{1}{2}$ and $|-\rangle$ with probability $\frac{1}{2}$"
    - Measure in the basis $(|+\rangle, |-\rangle)$, what do you obtain as distribution of outcomes?
    - Measure in the basis $(|0\rangle, |1\rangle)$, what do you obtain as distribution of outcomes?

It may be tempting to interpret: $\rho \overset{\text{def}}{=} \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ as "$|0\rangle$ with prob. $\frac{1}{2}$ and $|1\rangle$ with prob. $\frac{1}{2}$".

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Eigenvectors and eigenvalues of a density operator just indicates one of many possible sets that may give rise to a specific density matrix

What class of states does give rise to a particular density operator?

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What class of states does give rise to a particular density operator?

Theorem (admitted):

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i q_i |\varphi_i\rangle\langle\varphi_i|$ for quantum states $(|\psi_i\rangle)_i$ and $(|\varphi_i\rangle)_i$ and distributions $(p_i)_i$ and $(q_i)_i$ if and only if

$$\forall i, \quad \sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{i,j}\sqrt{q_j}\,|\varphi_j\rangle \quad \text{where } \mathbf{U} = (u_{i,j})_{i,j} \text{ be a unitary}$$

REDUCED DENSITY OPERATOR, PARTIAL TRACE

Aim:

Being able to describe the first qubit of the EPR-pair $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$

Problem:

Given $\rho^{AB} \in A \otimes B$,[a] what is the quantum state with respect to A?

[a] Abuse of notation, ρ density operator over $A \otimes B$

$$\longrightarrow \textit{Answer}: \rho^{A} \overset{\text{def}}{=} \mathbf{tr}_B\left(\rho^{AB}\right) \text{ where: } \begin{cases} \rho^{A} & \text{the reduced density operator for } A \\ \mathbf{tr}_B & \text{partial trace over } B \end{cases}$$

Definition: partial trace

Given $|a_1\rangle\langle a_2| \in A$ and $|b_1\rangle\langle b_2| \in B$, define

$$\mathbf{tr}_B\left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) = |a_1\rangle\langle a_2| \cdot \mathbf{tr}\left(|b_1\rangle\langle b_2|\right) = \langle b_1|b_2\rangle \cdot |a_1\rangle\langle a_2| \in A$$

then extend $\mathbf{tr}_B$ by linearity

We could have defined $\mathbf{tr}_B$ directly as:

$$\mathbf{tr}_B\left(\rho^{AB}\right) = \sum_i \left(\mathrm{Id} \otimes \langle i|\right)\rho^{AB}\left(\mathrm{I} \otimes |i\rangle\right) \quad \text{where } (|i\rangle)_i \text{ orthonormal basis of } B$$

But why this definition?

"Reduced density operator provides the correct measurement statistics for measurements made on system A"

Given an observable M on a A:

$\longrightarrow$ We want average measurements be the same when computed via ρ^A

or ρ^{AB} when we don't "act" on B, *i.e.*

$$\mathrm{tr}\left(\mathsf{M}\rho^A\right) = \mathrm{tr}\left((\mathsf{M}\otimes\mathsf{I})\,\rho^{AB}\right) \tag{1}$$

$\longrightarrow$ This equation is verified by $\rho^A = \mathrm{tr}_B\left(\rho^{AB}\right)$ $\Big($little exercise using $\mathrm{tr}(\mathsf{A}\otimes\mathsf{B}) = \mathrm{tr}(\mathsf{A})\,\mathrm{tr}(\mathsf{B})\Big)$

tr_B is the unique operator which verifies Equation (1):

Let f be a linear map of density operators on $A\otimes B$ to density operators on A which verifies the "average measurements"

$$\mathrm{tr}\left(\mathsf{M}\,f(\rho^{AB})\right) = \mathrm{tr}\left((\mathsf{M}\otimes\mathsf{I})\,\rho^{AB}\right)$$

Let M_i be an orthonormal basis to the space of Hermitian operators on A with respect to the scalar-product $\langle\mathsf{X},\mathsf{Y}\rangle = \mathrm{tr}\,(\mathsf{XY})$:

$$f(\rho^{AB}) = \sum_i \mathsf{M}_i\,\mathrm{tr}\left(\mathsf{M}_i\,f(\rho^{AB})\right) = \sum_i \mathsf{M}_i\,\mathrm{tr}\left((\mathsf{M}_i\otimes\mathsf{I})\,\rho^{AB}\right) = \sum_i \mathsf{M}_i\,\mathrm{tr}\left(\mathsf{M}_i\rho^A\right) = \rho^A$$

Therefore: any operator which verifies the "average measurements" is the partial trace!

Proposition:

Given two density operators ρ^A and ρ^B on a A and B:

$$\mathbf{tr}_B\left(\rho^A \otimes \rho^B\right) = \rho^A \quad \text{and} \quad \mathbf{tr}_A\left(\rho^A \otimes \rho^B\right) = \rho^B$$

$\mathbf{tr}_B$: “trace out B” ; $\mathbf{tr}_A$: “trace out A”

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tr_B: "trace out B" ; tr_A: "trace out A"

Proof:

Write $\rho_A = \sum_i \lambda_i |i\rangle\langle i|$ and $\rho_B = \sum_j \mu_j |j\rangle\langle j|$ (for orthonormal bases). By definition

$$\begin{aligned}
\mathrm{tr}_B\left(\rho^A \otimes \rho^B\right) &= \sum_{i,j} \lambda_i \mu_j \,\mathrm{tr}_B\left(|i\rangle\langle i| \otimes |j\rangle\langle j|\right) \\
&= \sum_{i,j} \lambda_i |i\rangle\langle i| \left(\sum_j \mu_j \langle j|j\rangle\right) \\
&= \rho^A
\end{aligned}$$

where in the last line we used $1 = \mathrm{tr}\left(\rho^B\right) = \sum_j \mu_j$

Consider the EPR-pair: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

1. Compute the density matrix ρ^{12} $\Big($1 and 2: first and second qubit$\Big)$ of the EPR-pair
2. Compute the reduced density matrices ρ^{1} and ρ^{2} with respect to the first and second qubit, respectively. What can you conclude?
3. Is $\rho^{12} = \rho^{1} \otimes \rho^{2}$?

Solution:

1. We have

$$\rho^{12} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

Therefore $\big($basis is ordered as $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)\big)$,

$$\rho^{12} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

2. Rewrite ρ^{12} as

$$\rho^{12} = \frac{1}{2}\left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\right)$$

Therefore,

$$\rho^{1} = \mathbf{tr}_2\left(\rho^{12}\right) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{\mathsf{Id}}{2} \quad \text{and} \quad \rho^{2} = \mathbf{tr}_1\left(\rho^{12}\right) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{\mathsf{Id}}{2}$$

Although the original system was prepared as a pure state $\big($complete knowledge$\big)$, the first and the second qubit are a uniform mixture of qubits!

3. No: $\rho^{12} \neq \rho^{1} \otimes \rho^{2} = \frac{\mathsf{Id}}{4}$

If Alice and Bob share an EPR-pair:

- Alice's qubit is a mixed state for which she has strictly no information/knowledge
- Bob's qubit is a mixed state for which he has strictly no information/knowledge

The joint state of the EPR pair is known exactly while both its first and second qubit is completely unknown (maximal uncertainty)!

Teleportation:

1. Recall that after Alice's measurement, the quantum state that Alice and Bob share is with probability $\frac{1}{4}$ the three-qubits state $|a, b\rangle \, |\psi_{ab}\rangle$:

$$|\psi_{ab}\rangle \stackrel{\text{def}}{=} \alpha \, |b\rangle + (-1)^a \beta \, |1 - b\rangle$$

   where $a, b \in \{0, 1\}$

   Compute the reduced density operator ρ_B of Bob's system $\Big($by tracing out the first two qubits$\Big)$ once Alice has performed her measurement but before Bob has learned a, b

2. What can you conclude?

1. We have the following computation:

$$\begin{aligned}\rho^{ab} &= |ab\rangle\langle ab| \otimes |\psi_{ab}\rangle\langle\psi_{ab}| \\ &= |ab\rangle\langle ab| \otimes \Big(|\alpha|^2 \, |b\rangle\langle b| + (-1)^a \alpha\overline{\beta} \, |b\rangle\langle 1-b| + \\ &\qquad (-1)^a \overline{\alpha}\beta \, |1-b\rangle\langle b| + |\beta|^2 \, |1-b\rangle\langle 1-b| \Big)\end{aligned}$$

The density operator of the shared quantum state is:

$$\rho = \frac{1}{4}\left(\sum_{a,b\in\{0,1\}} \rho^{ab}\right)$$

By tracing out the first two qubits we get

$$\begin{aligned}\rho_{\mathrm{B}} &= \frac{1}{4}\left(\left(2|\alpha|^2 + 2|\beta|^2\right)|0\rangle\langle 0| + \left(2|\alpha|^2 + 2|\beta|^2\right)|1\rangle\langle 1|\right) \\ &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \\ &= \frac{\mathbf{Id}}{2}\end{aligned}$$

2. Bob's state has no dependence upon the state $|\psi\rangle$ being teleported: any measurements performed by Bob will contain no information about $|\psi\rangle$. It prevents Alice to transmit information to Bob faster than light!

LET ME INSIST

Whatever Alice is doing on her qubit, Bob has a uniform mixture of qubits

$$\rho = \frac{\mathsf{Id}}{2}$$

- Whatever the unitary evolution,
$$U\rho U^{\dagger} = \rho = \frac{\mathsf{Id}}{2}$$
- Whatever is the applied measure $(M_m)_m$

$$p(m) = \mathrm{tr}\left(M_m^{\dagger} M_m \rho\right) = \mathrm{tr}\left(M_m^{\dagger} M_m \frac{\mathsf{Id}}{2}\right) = \frac{1}{2}\,\mathrm{tr}\left(M_m^{\dagger} M_m\right)$$

There are no dependence in $|\psi\rangle$! Bob cannot extract anything about $|\psi\rangle$!

REMEMBER

Whatever Alice is doing on her own, Bob has a uniform mixture of qubits

$\longrightarrow$ Bob cannot do anything to recover $|\psi\rangle$!

Except if Bob knows Alice's measurement

$\longrightarrow$ The fact that information cannot be shared faster than light is prevented!

SCHMIDT DECOMPOSITION AND PURIFICATION

Density operators and partial trace:

$\longrightarrow$ Useful for studying composite quantum systems!

Two new useful tools:

- Schmidt decomposition
- Purification

Theorem: Schmidt decomposition (admitted):

For any pure $|\psi\rangle \in A \otimes B$, it exists

- a **unique integer** d
- an orthonormal set $|a_1\rangle, \ldots, |a_d\rangle \in A$
- an orthonormal set $|b_1\rangle, \ldots, |b_d\rangle \in B$
- $\lambda_1, \ldots, \lambda_d > 0$

such that

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |a_i\rangle |b_i\rangle$$

First consequence:

Given (pure) $|\psi\rangle \in A \otimes B$, then

$$\rho^A = \mathrm{tr}_B(|\psi\rangle\langle\psi|) = \sum_{i=1}^{d} \lambda_i^2 |a_i\rangle\langle a_i| \quad \text{and} \quad \rho^B = \mathrm{tr}_A(|\psi\rangle\langle\psi|) = \sum_{i=1}^{d} \lambda_i^2 |b_i\rangle\langle b_i|$$

Therefore, **ρ^A and ρ^B have the same eigenvalues: the λ_i^2's and possibly 0!**

Definition: Schmidt's number

Given pure $|\psi\rangle \in A \otimes B$ with Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |a_i\rangle |b_i\rangle$$

The integer d **is called Schmidt number**. This number does not depend on the decomposition and it depends only on $|\psi\rangle$

Theorem: a useful characterization of entanglement

A pure state $|\psi\rangle \in A \otimes B$ is entangled if and only if its Schmidt's number is > 1 if and only if ρ_A and ρ_B are mixed states $\left(\text{where } \rho = |\psi\rangle\langle\psi|\right)$

Proof:

See Exercise Session

Given a mixed state ρ of A: is it possible to introduce another system R and a pure state $|\psi\rangle \in A \otimes R$ such that

$$\rho = \mathrm{tr}_R \left(|\psi\rangle\langle\psi| \right)$$

Yes!

Spectral decomposition in an orthonormal basis of ρ:

$$\rho = \sum_{i=1}^{n} \lambda_i \, |i\rangle\langle i| \quad \left(\text{the } \lambda_i\text{'s are} \geq 0\right)$$

It is enough (exercise!) to define $|\psi\rangle$ as:

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{\lambda_i} \, |i\rangle \, |i\rangle$$

$\longrightarrow$ This process is known as purification!

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$\longrightarrow$ This process is known as purification!

Relation between Schmidt decomposition and purification:

Purifying a mixed state: define a pure state whose Schmidt basis is just the basis in which the mixed state is diagonal!

When designing $\big($advanced$\big)$ quantum algorithms it may happen that

- we want to take into account in a clean way measurements when sophisticated quantum entanglement is at stake

  $\longrightarrow$ Density operator formalism!

- we want to forget some qubits

  $\longrightarrow$ Density operator formalism with partial trace!

- we want pure states although some measurements have been performed

  $\longrightarrow$ Density operator formalism with purification at the cost of adding ancillas qubits!

However, in many situations, density operator is not useful, it adds useless formalism

If you are interested in information theory and quantum information theory:

One book chapter about this topic $\Big($to be presented at the end of the course$\Big)$

- Chapter 11 up to 12.3 in *Quantum Computation and Quantum Information*, Michael A. Nielsen and Isaac L. Chuang

Be careful:

I advise to follow lectures about classical information theory for this presentation!

EXERCISE SESSION