LECTURE 5
GROVER'S SEARCH ALGORITHM AND INTRODUCTION TO THE QUANTUM FOURIER TRANSFORM

INF587 Quantum computer science and applications

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THE OBJECTIVE OF THE DAY

- Grover's algorithm
- Introduction to the Quantum Fourier Transform (QFT) but by starting with the classical case!

1. Grover's search algorithm
2. Amplitude amplification
3. Introduction to the discrete Fourier transform
4. Quantum Fourier Transform (QFT) over $\mathbb{Z}/2^n\mathbb{Z}$: $\mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}}$

GROVER'S SEARCH ALGORITHM

Given some list L, what is the cost for classically finding a fixed x_0?

$\longrightarrow$ It is, a priori, $\sharp L$!

But is it always the case?

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$\longrightarrow$ It is, a priori, $\sharp L$!

But is it always the case? No!

If the list L has some “structure” it can be helpful:

- Sorted list: time $\log \sharp L$ with a dichotomic search,
- Hash table: constant time.

Our aim with Grover’s algorithm: treating quantumly the case where we are given a list without any structure

Search problem

- **Input**: a function $f : \{0,1\}^n \longrightarrow \{0,1\}$,
- **Goal**: find $\mathbf{x} \in \{0,1\}^n$ be such that $f(\mathbf{x}) = 1$.

$\longrightarrow$ Can be viewed as a modelling of a data search in an unstructured database $(x, f(\mathbf{x}))_{\mathbf{x} \in \{0,1\}^n}$ of size 2^n (exponential)

Finding a solution: let $t \stackrel{\text{def}}{=} \sharp \{\mathbf{x} \in \{0,1\}^n : f(\mathbf{x}) = 1\}$

Let $N = \sharp\{0,1\}^n = 2^n$.

- Classically a randomized algorithm would need $\Theta\left(\frac{N}{t}\right)$ queries to f and in time $O\left(\frac{N}{t}\ \mathrm{Cost}(f)\right)$
- Grover can solve this problem with only $O\left(\sqrt{\frac{N}{t}}\right)$ queries to f and in time $O\left(\sqrt{\frac{N}{t}}\ \mathrm{Cost}(f)\right)$

Symmetric cryptography: exhaustive search of the secret key with 128 bits in AES (encryption) requires 2^{128} classical operations

$\longrightarrow$ Quantumly: 2^{64} operations which is reachable...

Consequence:

$\longrightarrow$ All secret keys in symmetric encryption have to be size $\times 2$ (at least...)

Grover offers a generic attack against symmetric encryption schemes, but there are many other ways of taking advantage of quantum computers...

- *Quantum Attacks without Superposition Queries: the Offline Simon's Algorithm.* X. Bonnetain, A. Hosoyamada, M. Naya-Plasencia, Y. Sasaki, A. Schrottenloher: https://eprint.iacr.org/2019/614.pdf

Lower-bound

Any algorithm solving the search problem for $f : \{0,1\}^n \longrightarrow \{0,1\}$ with t solutions needs to make

$$\Omega\left(\sqrt{\frac{2^n}{t}}\right) \text{ queries to } f$$

$\longrightarrow$ **Grover's algorithm is "optimal"** (up to constants) in the number of queries to f

A good/bad news

If the Grover's search problem was solvable in time $\log^c 2^n$: any NP-problem could be solvable (with good probability) in polynomial time with a quantum computer...

$\longrightarrow$ There are lower-bounds for the running time of quantum algorithms solving some problems!

- *Lecture notes by Ronald de Wolf's* , *Chapters* 11.

  https://arxiv.org/pdf/1907.09415.pdf

First, with quantum parallelism, we build:

$$|\psi\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle\, |f(x)\rangle$$

(I) Fundamental idea of Grover algorithm

Write $|\psi\rangle$ as:

$$|\psi\rangle = \sin\theta \, |\psi_{\text{good}}\rangle + \cos\theta \, |\psi_{\text{bad}}\rangle \quad \text{where} \quad \begin{cases} |\psi_{\text{good}}\rangle = \frac{1}{\sqrt{t}} \sum\limits_{\substack{x \in \{0,1\}^n \\ f(x)=1}} |x\rangle\, |f(x)\rangle \\ |\psi_{\text{bad}}\rangle = \frac{1}{\sqrt{2^n - t}} \sum\limits_{\substack{x \in \{0,1\}^n \\ f(x)=0}} |x\rangle\, |f(x)\rangle \end{cases}$$

with $|\psi_{\text{good}}\rangle$ and $|\psi_{\text{bad}}\rangle$ are quantum states by definition of t (number of solutions).

But what is the value of θ?

IDEA: SPLIT YOUR QUANTUM STATE

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But what is the value of θ?

$\longrightarrow$ θ is such that $\frac{\sin\theta}{\sqrt{t}} = \frac{1}{\sqrt{2^n}} \iff \theta = \arcsin\sqrt{\frac{t}{2^n}}$ (we need to know t to know θ)

(II) Fundamental idea of Grover algorithm

Move θ to $\frac{\pi}{2}$!

THE ANGLE θ?

$|\psi\rangle = \sin\theta \left|\psi_{\text{good}}\right\rangle + \cos\theta \left|\psi_{\text{bad}}\right\rangle$ where $\left|\psi_{\text{good}}\right\rangle$ uniform superposition of solutions

How is θ when there is few solutions, namely $t \ll 2^n$?

THE ANGLE θ?

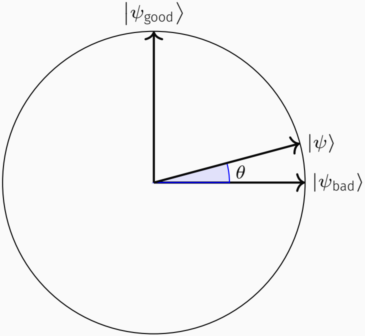
$$|\psi\rangle = \sin\theta \left|\psi_{\text{good}}\right\rangle + \cos\theta \left|\psi_{\text{bad}}\right\rangle \quad \text{where } \left|\psi_{\text{good}}\right\rangle \text{ uniform superposition of solutions}$$

How is θ when there is few solutions, namely $t \ll 2^n$?

$$\longrightarrow \sin\theta = \sqrt{\tfrac{t}{2^n}}, \text{ therefore } \theta \approx \sqrt{\tfrac{t}{2^n}} \approx 0$$

Exercise Session 4: we can make reflexions over a quantum state!

We start by building $|\psi\rangle$



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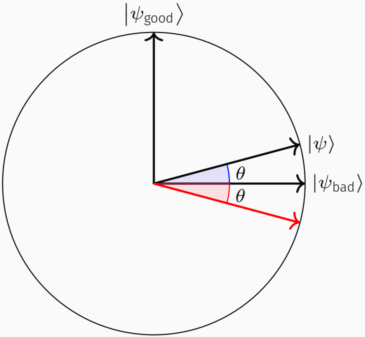
Reflexion over $|\psi_{\text{bad}}\rangle$



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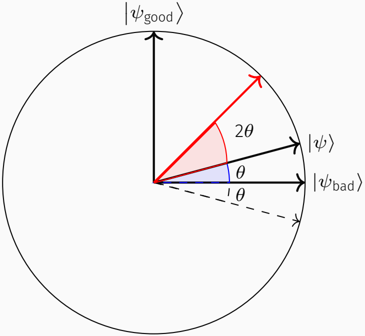

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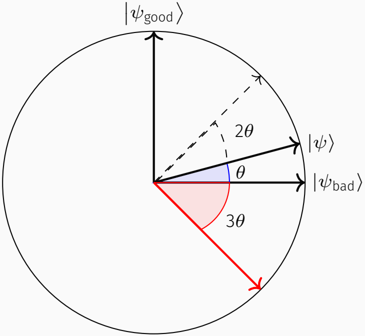

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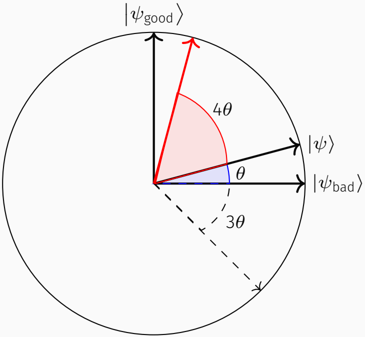
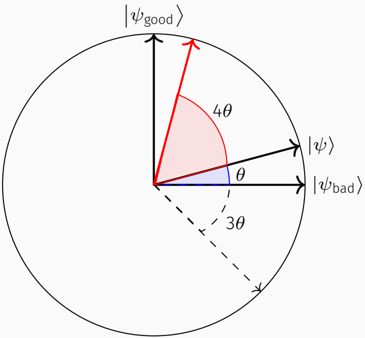


Exercise Session 4: we can make reflexions over a quantum state!

and so on up to $\pi/2$...



Number k of iterations to reach $|\psi_{\text{good}}\rangle$: $\theta \to (2k+1)\theta$

Choose the number k of iterations (reflexions over $|\psi_{\text{bad}}\rangle$ and $|\psi\rangle$) such that

$$(2k+1)\theta = \frac{\pi}{2} \iff k = \frac{\pi}{4\theta} - 1 = \frac{\pi}{4 \arcsin \sqrt{\frac{t}{2^n}}} - 1 \approx \frac{\pi}{4}\sqrt{\frac{2^n}{t}}$$

$$|\psi_{\text{good}}\rangle = \frac{1}{\sqrt{t}} \sum_{\substack{x \in \{0,1\}^n \\ f(x)=1}} |x\rangle |f(x)\rangle \quad \text{and} \quad |\psi_{\text{bad}}\rangle = \frac{1}{\sqrt{2^n - t}} \sum_{\substack{x \in \{0,1\}^n \\ f(x)=0}} |x\rangle |f(x)\rangle$$

Reflexion $R_{|\psi_{\text{bad}}\rangle}$ over $|\psi_{\text{bad}}\rangle$:

$$I_n \otimes Z : |x\rangle |b\rangle \longmapsto (-1)^b |x\rangle |b\rangle$$

Reflexion $R_{|\psi\rangle}$ over $|\psi\rangle$

Exercise session 4: we can build a reflexion $\mathbf{R}_{|\psi\rangle}$ over $|\psi\rangle$ with $O(n)$ elementary gates and two calls to $\mathbf{U}$ which is such that

$$\mathbf{U}\,|0^n\rangle |0\rangle = |\psi\rangle \; \left(= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle \right)$$

$$\longrightarrow \text{Choose } \mathbf{U} = \mathbf{U}_f \left(\mathbf{H}^{\otimes n} \otimes \mathbf{I}_2 \right)$$

$\longrightarrow$ In Grover's algorithm we crucially used that $|\psi\rangle$ **can be built**!

Proposition

We have:

$$\cos\alpha\,|\psi_{\text{bad}}\rangle + \sin\alpha\,|\psi_{\text{good}}\rangle \xrightarrow{\mathsf{R}_{|\psi\rangle}\mathsf{R}_{|\psi_{\text{bad}}\rangle}} \cos(2\theta+\alpha)\,|\psi_{\text{bad}}\rangle + \sin(2\theta+\alpha)\,|\psi_{\text{good}}\rangle$$

Proof:

$$|\psi\rangle = \cos\theta\,|\psi_{\text{bad}}\rangle + \sin\theta\,|\psi_{\text{good}}\rangle \;\perp\; |\psi^{\perp}\rangle = \sin\theta\,|\psi_{\text{bad}}\rangle - \cos\theta\,|\psi_{\text{good}}\rangle$$

From there:

$$|\psi_{\text{bad}}\rangle = \cos\theta\,|\psi\rangle + \sin\theta\,|\psi^{\perp}\rangle \quad \text{and} \quad |\psi_{\text{good}}\rangle = \sin\theta\,|\psi\rangle - \cos\theta\,|\psi^{\perp}\rangle$$

By definition of the reflexions and trigonometric rules:

$$\begin{aligned}
&\mathsf{R}_{|\psi\rangle}\mathsf{R}_{|\psi_{\text{bad}}\rangle}\left(\cos\alpha\,|\psi_{\text{bad}}\rangle + \sin\alpha\,|\psi_{\text{good}}\rangle\right) = \mathsf{R}_{|\psi\rangle}\left(\cos\alpha\,|\psi_{\text{bad}}\rangle - \sin\alpha\,|\psi_{\text{good}}\rangle\right)\\
&= \mathsf{R}_{|\psi\rangle}\left(\cos\alpha\cos\theta - \sin\alpha\sin\theta\right)|\psi\rangle + \left(\cos\alpha\sin\theta + \sin\alpha\cos\theta\right)|\psi^{\perp}\rangle\\
&= \cos(\alpha+\theta)\,|\psi\rangle - \sin(\alpha+\theta)\,|\psi^{\perp}\rangle\\
&= \left(\cos(\alpha+\theta)\cos\theta - \sin\alpha\sin(\theta+\alpha)\right)|\psi_{\text{bad}}\rangle + \left(\cos(\alpha+\theta)\sin\theta + \sin(\alpha+\theta)\cos\theta\right)|\psi_{\text{good}}\rangle\\
&= \cos(2\theta+\alpha)\,|\psi_{\text{bad}}\rangle + \sin(2\theta+\alpha)\,|\psi_{\text{good}}\rangle \qquad \square
\end{aligned}$$

Grover's algorithm

1. Build $|\psi\rangle = \cos\theta\,|\psi_{\text{bad}}\rangle + \sin\theta\,|\psi_{\text{good}}\rangle$,
2. Apply k **times** the unitary $\mathsf{R}_{|\psi\rangle}\mathsf{R}_{|\psi_{\text{bad}}\rangle}$ on the quantum state $|\psi\rangle$.
3. Measure, if the last qubit is 1 return the first n qubits; otherwise repeat from step 1.

Probability of success (use the previous proposition)

$$P_k = \sin^2(2k\theta + \theta)$$

$\longrightarrow$ How to choose the **number of iterations k**?

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$\longrightarrow$ How to choose the **number of iterations k**?

Choose $k \stackrel{\text{def}}{=} \left\lceil \left(\frac{\pi}{2} - \theta\right) \right\rceil \frac{1}{2\theta}$, then (again some calculations)

$$P_k \geq \frac{1}{4} \quad \text{and } k = O\left(\sqrt{\frac{2^n}{t}}\right) \text{ as } \theta = \arcsin\sqrt{\frac{t}{2^n}}$$

Grover's algorithm finds a solution with constant probability by running $O\left(\sqrt{\frac{2^n}{t}}\right)$ the unitary $\mathsf{R}_{|\psi\rangle}\mathsf{R}_{|\psi_{\text{bad}}\rangle}$.

- $\mathsf{R}_{|\psi_{\text{bad}}\rangle} = \mathsf{I}_n \otimes \mathsf{Z}$: one quantum gate
- $\mathsf{R}_{|\psi\rangle}$: $O(n)$ quantum gates + 2 calls to $\mathsf{U} = \mathsf{U}_f\left(\mathsf{H}^{\otimes n} \otimes \mathsf{I}_2\right)$

Cost of Grover's algorithm

The cost of Grover's algorithm to find a solution, with constant probability, in the quantum gate model is given by

$$O\left(\sqrt{\frac{2^n}{t}}\max(n, T_f)\right)$$

where T_f is the classical running time to compute f.

- We need to run the algorithm $\left\lceil \left(\frac{\pi}{2} - \theta\right) \right\rceil \frac{1}{2\theta}$ where $\theta = \arcsin \sqrt{\frac{t}{2^n}}$ and therefore to know t...

  $\longrightarrow$ If iterations chosen too big, the success probability $\sin((2k+1))^2$ goes down!

- if t is known, can we tweak the algorithm to end up in exactly the good state, namely $P_k = 1$?

$\longrightarrow$ See the exercise session to overcome these issues!

AMPLITUDE AMPLIFICATION

$\mathcal{A}$ be a classical/quantum algorithm that can find a solution x (*i.e.*, $f(\mathsf{x}) = 1$) with probability p

$\longrightarrow$ One can repeat $O\left(\frac{1}{p}\right)$ times $\mathcal{A}$ to find a solution with constant probability.

Why?

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Why?

Amplitude Amplification

Assume you have a classical or quantum algorithm $\mathcal{A}$ (**without measurement**) that can find a solution $\mathbf{x}$ to the search problem in time T with probability p.

If f is computable in time T_f, then we can compute (quantumly) a solution in time $O\left(\frac{T}{\sqrt{p}} \max(n, T_f)\right)$ with constant probability.

Pick a random $\mathbf{x} \in \{0,1\}^n$ and output $\mathbf{x}$

$\longrightarrow$ This algorithm runs in time $O(n)$ and it finds a solution with probability $p = \frac{t}{2^n}$.

Using amplitude amplification: you can find a solution in time $\approx \sqrt{\frac{2^n}{t}}$

Grover may be seem as the quantization of the random search in an unstructured data set...

$\longrightarrow$ Amplitude amplification is more useful:

sometimes, we know algorithms better than random search and amplitude amplification shows we also have a quadratic speedup!

Lecture 4:

If $\mathcal{A}$ is quantum: measurements only at the end of the computation and starts from $\left|0^m\right\rangle$

$\longrightarrow$ Before the final measurement: $\mathcal{A}$ outputs a state $|\psi\rangle$, and measuring the output register gives a solution $\mathbf{x}$ with probability p.

$$\mathcal{A}\left|0^m\right\rangle = |\psi\rangle = \sum_{\mathsf{x}\in\{0,1\}^n} \alpha_{\mathsf{x}} |\mathsf{x}\rangle |\varphi_{\mathsf{x}}\rangle\ , \text{ where } \sum_{\mathsf{x}:f(\mathsf{x})=1} |\alpha_{\mathsf{x}}|^2 = p.$$

Write:

$$|\psi\rangle = \sin\theta\left|\psi_{\text{good}}\right\rangle + \cos\theta\left|\psi_{\text{bad}}\right\rangle \quad \text{where } \left|\psi_{\text{good}}\right\rangle \stackrel{\text{def}}{=} \frac{1}{\sin\theta}\sum_{\substack{\mathsf{x}\in\{0,1\}^n\\ f(\mathsf{x})=1}} \alpha_{\mathsf{x}} |\mathsf{x}\rangle |\alpha_{\mathsf{x}}\rangle$$

$$\text{where } \sin\theta = \sqrt{p}$$

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$$\text{where } \sin\theta = \sqrt{p}$$

Run Grover's algorithm with the reflexions $\mathsf{R}_{|\psi_{\text{bad}}\rangle} : |\mathsf{x}\rangle |\mathsf{y}\rangle \mapsto (-1)^{f(\mathsf{x})} |\mathsf{x}\rangle |\mathsf{y}\rangle$ (see Exercise Session to compute this unitary) and $\mathsf{R}_{|\psi\rangle}$ over $|\psi\rangle$ but:

$$\mathsf{R}_{|\psi\rangle} \neq O(n) \text{ quantum gates + 2 calls to } \mathsf{U} = \mathsf{U}_f\left(\mathsf{H}^n \otimes \mathsf{I}_2\right) \text{ which was designed to build } \frac{1}{\sqrt{2^n}}\sum_{\mathsf{x}} |\mathsf{x}\rangle |f(\mathsf{x})\rangle\ldots$$

Amplitude amplification: $\mathsf{R}_{|\psi\rangle}$ is $O(n)$ quantum gates + 1 call to $\mathsf{U} = \mathcal{A}$ and 1 call to $\mathsf{U}^{-1} = \mathcal{A}^{-1}$

BE CAREFUL

When performing amplitude amplification on a quantum algorithm $\mathcal{A}$, we supposed it performs no measurements (at least we restrict $\mathcal{A}$ before its final measurement)

$\longrightarrow$ to be able to perform $\mathcal{A}^{-1}$

Grover's search algorithm in amplitude amplification shows a strong statement. Given

$$|\psi\rangle = \alpha |\psi_V\rangle + \beta \left|\psi_V^\perp\right\rangle \text{ where } |\psi_V\rangle \in \mathrm{Span}\left(|x\rangle\ : f(x) = 1\right) \text{ and } \left|\psi_V^\perp\right\rangle \in \mathrm{Span}\left(|x\rangle\ : f(x) = 1\right)^\perp$$

After amplitude amplification: $\left|\psi'\right\rangle \approx |\psi_V\rangle$

(even equal with exact grover when amplitude α is known)

Be careful:

To run amplitude amplification: you need to be able to build $|\psi\rangle$...

Lecture 4: given a deterministic $\mathcal{A}$, one can run $\mathsf{U}_{\mathcal{A}}$ in $\approx$ same time

If $\mathcal{A}$ is randomized?

Classical modelization (think R be the seed of a pseudo-random generator):

$$\mathcal{A} : \text{pick a random } \mathsf{R} \in \{0,1\}^r, \text{ compute } \mathcal{A}(\mathsf{R}) \text{ to get some outcome } \mathsf{x}_\mathsf{R}.$$

$\longrightarrow$ Randomness chosen at the beginning: the algorithm can be interpreted as deterministic.

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$$\mathsf{U}_{\mathcal{A}}(|\mathsf{R}\rangle|\mathsf{y}\rangle) = |\mathsf{R}\rangle|\mathsf{y}+\mathsf{x}_\mathsf{R}\rangle\,.$$

$$|0^r\rangle|0^n\rangle \xrightarrow{\mathsf{H}^{\otimes r}\otimes \mathsf{I}_r} \frac{1}{\sqrt{2^r}}\sum_{\mathsf{R}\in\{0,1\}^r}|\mathsf{R}\rangle|0^n\rangle \xrightarrow{\mathsf{U}_{\mathcal{A}}} \frac{1}{\sqrt{2^r}}\sum_{\mathsf{R}\in\{0,1\}^r}|\mathsf{R}\rangle|\mathsf{x}_\mathsf{R}\rangle\,.$$

measuring outputs a solution with probability p.

$\longrightarrow$ We can use amplitude amplification on this algorithm!

DISCRETE FOURIER TRANSFORM

- $(G,+)$ be a finite Abelian group
- Character group: $\widehat{G} = \{\chi_g : g \in G\} \cong G$, set of characters = homomorphism from G to the unit complex circle $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$

$$\begin{array}{rcl} \chi_g : G & \longrightarrow & \mathbb{U} \\ x & \longmapsto & \chi_g(x), \text{ such that} \end{array}$$

$$\forall x, y \in G,\ \chi_g(x+y) = \chi_g(x)\chi_g(y)$$

Examples:

- $G = \mathbb{F}_2^n = \underbrace{\mathbb{F}_2 \times \cdot \times \mathbb{F}_2}_{n \text{ times}}$,[a]

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n, \quad \chi_{\mathbf{x}}(\mathbf{y}) = (-1)^{\mathbf{x}\cdot\mathbf{y}} \quad \text{where } \mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

- $G = \mathbb{Z}/2^n\mathbb{Z}$,

$$\forall x, y \in \mathbb{Z}/2^n\mathbb{Z}, \quad \chi_x(y) = e^{-\frac{2i\pi xy}{2^n}}$$

[a] $\mathbb{F}_2$ binary field, $\{0,1\}$ embedded with $\oplus$

Nice reading about characters on finite Abelian groups:

https://kconrad.math.uconn.edu/blurbs/grouptheory/charthy.pdf

$$\sum_{g\in G}\chi_x(g)\overline{\chi_y}(g) = \begin{cases} \sharp G & \text{if } \chi_x = \chi_y \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \sum_{\chi\in\widehat{G}}\chi(x)\overline{\chi}(y) = \begin{cases} \sharp G & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

- The matrix $\left(\frac{\chi_x(y)}{\sqrt{\sharp G}}\right)_{x,y\in G}$ is unitary, in particular:

$$\left(\frac{\chi_x}{\sqrt{\sharp G}}\right)_{x\in G} \text{ orthonormal basis for the scalar product } \langle f,g\rangle = \sum_{y\in G} f(y)\overline{g}(y)$$

$\left(\frac{\chi_x}{\sqrt{\sharp G}}\right)_{x\in G}$ sometimes called the "Fourier basis".

- The translation operator is diagonal in the Fourier basis

$$\begin{array}{rcl} \tau_a : (G\to\mathbb{C}) & \longrightarrow & (G\to\mathbb{C}) \\ f & \longmapsto & \tau_a(f) : x\in G\mapsto f(x+a) \text{ then} \end{array}$$

$$\tau_x(\chi_y) = \underbrace{\chi_y(a)}_{\text{egeinvalue}} \underbrace{\chi_y}_{\text{eigenvector}}$$

Exercise:

1. Prove that for any character $\chi \in \widehat{G}$,

$$\sum_{g\in G} \chi(g) = \begin{cases} \sharp G & \text{if } \chi = 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. How do you deduce from that

$$\sum_{g\in G} \chi_x(g)\overline{\chi_y}(g) = \begin{cases} \sharp G & \text{if } \chi_x = \chi_y \\ 0 & \text{otherwise.} \end{cases}$$

3. Consider the function f_g

$$\begin{array}{rcl} f_g : \widehat{G} & \longrightarrow & G \\ \chi & \longmapsto & \chi(g), \text{ such that} \end{array}$$

   What can you say about f_g?

4. How can you deduce from the previous point that we also have

$$\sum_{\chi\in \widehat{G}} \chi(x)\overline{\chi}(y) = \begin{cases} \sharp G & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Orthogonal subgroup

For a subgroup H of G we denote by $H^\perp$ the orthogonal subgroup defined by

$$H^\perp \stackrel{\text{def}}{=} \{g \in G \ : \ \forall h \in H, \ \chi_g(h) = 1\}$$

$\longrightarrow$ Important concept in Simon/Shor's algorithm! (see Lecture 6)

$$\sum_{h \in H} \chi_g(h) = \begin{cases} \sharp H & \text{if } g \in H^\perp \\ 0 & \text{otherwise.} \end{cases}$$

Fourier transform

Given a finite abelian group G and $f : G \longrightarrow \mathbb{C}$, its Fourier transform is

$$\forall x \in G, \quad \widehat{f}(x) = \frac{1}{\sqrt{\sharp G}} \sum_{y \in G} f(y)\overline{\chi_x}(y)$$

Notice that:

$$\widehat{f}(x) = \left\langle f, \frac{\chi_x}{\sqrt{\sharp G}} \right\rangle \text{ where standard scalar product } \langle \cdot, \cdot \rangle \text{ over functions } G \longrightarrow \mathbb{C}$$

$\left(\frac{\chi_x}{\sqrt{\sharp G}} \right)_{x \in G}$ orthonormal basis for this scalar product and $\widehat{f}(x)$: x-thm coefficient of f in this basis.

Exercise

Compute the Fourier transform of the following function $\mathbb{F}_2^n \longrightarrow \mathbb{C}$,

- $f(\mathbf{0}) = 1$ and 0 otherwise.
- $\forall \mathbf{x} \in \mathbb{F}_2^n, f(\mathbf{x}) = \frac{1}{2^n}$.
- Does it remind you something?

Classical Fourier Transform	Quantum Fourier Transform: QFT_G
$f = (f(x))_{x\in G}$	$\lvert\psi_f\rangle = \sum_{x\in G} f(x)\lvert x\rangle \ (\lVert f\rVert_2 = 1)$
$\widehat{f}(x) = \frac{1}{\sqrt{\sharp G}}\sum_{y\in G} f(y)\overline{\chi_x}(y)$	$\mathrm{QFT}_G\lvert\psi\rangle \stackrel{\text{def}}{=} \widehat{\lvert\psi_f\rangle} = \sum_{x\in G}\widehat{f}(x)\lvert x\rangle$

$\longrightarrow$ In particular: $\forall x \in G$, **QFT_G** $\lvert x\rangle = \frac{1}{\sqrt{\sharp G}}\sum_{y\in G}\overline{\chi_y}(x)\lvert y\rangle$

$\Big($ It corresponds to the fact that $\widehat{\delta_x}(y) = \frac{1}{\sqrt{\sharp G}}\overline{\chi_y}(x)$ where δ_x Kronecker symbol and δ_x"$=$"$\lvert x\rangle$ $\Big)$

Exercise:

Show that $\lvert\psi_f\rangle$ is a quantum state.

Formally, given any finite group G: $(\lvert x\rangle)_{x\in G}$ denotes an orthonormal basis of an Hilbert space of dimension $\sharp G$.

Given $\mathbf{x}$, what is the cost for (classically) computing $\widehat{f}(x)$?

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$\longrightarrow$ It costs $\sharp G$... Be careful: in practice $\sharp G = 2^n$

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What is the cost for (classically) computing $\widehat{f}$, namely all the $\widehat{f}(x)$'s?

$\longrightarrow$ It costs naively $\sharp G^2$.

$\longrightarrow$ We can do much better to compute $\widehat{f}$

The Fast Fourier Transform (FFT): computing $\widehat{f}$ costs $O(\sharp G \log \sharp G)$ (in most cases...)

Suppose that $G = \mathbb{Z}/2^n\mathbb{Z}$, in particular $\sharp G = 2^n$

Let $N \overset{\text{def}}{=} 2^n$ and $\omega_N = e^{-\frac{2i\pi}{N}}$.

Divide and Conquer strategy

$$\begin{aligned}
\widehat{f}(j) &= \frac{1}{\sqrt{2^n}} \sum_{k=0}^{N-1} f(k)\omega_N^{-jk} \\
&= \frac{1}{\sqrt{N}} \left(\sum_{k \text{ even}} f(k)\omega_N^{-jk} + \omega_N^{-j} \sum_{k \text{ odd}} f(k)\omega_N^{-j(k-1)} \right) \\
&= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{N/2}} \sum_{k \text{ even}} f(k)\omega_{N/2}^{-j/2k} + \omega_N^{-j} \frac{1}{\sqrt{N/2}} \sum_{k \text{ odd}} f(k)\omega_{N/2}^{-j/2(k-1)} \right)
\end{aligned}$$

$\longrightarrow$ Therefore we reduce the computation of $\widehat{f}(j)$ to two Fourier transforms over $\mathbb{Z}/2^{n-1}\mathbb{Z}$

Cost: $T(2^n) = 2T(2^{n-1}) + O(2^n)$, therefore $T(2^n) = O(2^n \underbrace{\log(2^n)}_{\text{rec. calls}}) = O(n2^n)$

Computing the quantum Fourier transform

- QFT_G can be implemented by a quantum circuit of size $O\left(\log^3 \sharp G\right)$ for any arbitrary finite Abelian group G.
- $\mathbf{QFT}_{\mathbb{Z}/N\mathbb{Z}}$ can be implemented by a quantum circuit of size $O\left(\log^3 N\right)$
- $\mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}}$ can be implemented by a quantum circuit of size $O\left(n^2\right)$ (here $n = \log 2^n = \log \sharp\mathbb{Z}/2^n\mathbb{Z}$).
- $\mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}}$ can be implemented up to some accuracy [a] by a quantum circuit of size $O\left(n \log n\right)$.

[a] for the norm operator

$\longrightarrow$ Exponentially faster than computing the classical Fourier transform, even with the FFT trick which is for instance $O(n2^n)$ in the case of $\mathbb{Z}/2^n\mathbb{Z}$.

Quantum Fourier over $\mathbb{F}_2^n$ (the set $\{0,1\}^n$ with the $\oplus$ operation term by term)?

$\longrightarrow$ Characters are given by $\chi_{\mathbf{x}}(\mathbf{y}) = (-1)^{\mathbf{x}\cdot\mathbf{y}}$ where $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^n x_i y_i$.

$$\widehat{f}(\mathbf{x}) = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{y}\in\mathbb{F}_2^n} (-1)^{\mathbf{x}\cdot\mathbf{y}} f(\mathbf{y})$$

Quantum Fourier transform in $\mathbb{F}_2^n$

$$\mathrm{QFT}_{\mathbb{F}_2^n} |\mathbf{x}\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{y}\in\mathbb{F}_2^n} (-1)^{\mathbf{x}\cdot\mathbf{y}} |\mathbf{y}\rangle$$

$\longrightarrow$ $\mathrm{QFT}_{\mathbb{F}_2^n} = \mathsf{H}^{\otimes n}$ and its **cost:** $O(n)$

QUANTUM FOURIER TRANSFORM $\mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}}$

Give an efficient quantum circuit for computing $\mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}}$

Gates that we will use:

$$\mathsf{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ (Hadamard)} \qquad \mathsf{R}_s = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2i\pi}{2^s}} \end{pmatrix} \text{ (Phase rotation)}$$

$$\mathsf{C}\text{-}\mathsf{R}_s : \begin{cases} |0\rangle\,|x\rangle \mapsto |0\rangle\,|x\rangle \\ |1\rangle\,|x\rangle \mapsto |1\rangle\,\mathsf{R}_s\,|x\rangle \end{cases} \quad \text{(Controlled-}\mathsf{R}_s\text{)}$$

Notation:

For any integer $j \in [\![0, 2^n - 1]\!]$, binary decomposition $j = j_1, \ldots, j_n$ where j_1 bit of highest weight

$$j = \sum_{\ell=1}^{n} 2^{n-\ell} j_\ell$$

For any $x \in [\![0, 2^n - 1]\!]$,

$$|x\rangle = |x_1, \ldots, x_n\rangle$$

$$\begin{aligned}
\mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}} |k\rangle &= \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{\frac{2ikj\pi}{2^n}} |j\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{2i\pi k\left(\sum_{\ell=1}^{n} 2^{-\ell} j_\ell\right)} |j_1, \ldots, j_n\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \prod_{\ell=1}^{n} e^{2i\pi k 2^{-\ell} j_\ell} |j_1, \ldots, j_n\rangle \\
&= \bigotimes_{\ell=1}^{n} \left(\frac{|0\rangle + e^{2i\pi k 2^{-\ell}} |1\rangle}{\sqrt{2}} \right)
\end{aligned}$$

$\longrightarrow \mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}} |k\rangle$ is a separable quantum state!

Be careful: we crucially use that we work in $\mathbb{Z}/2^n\mathbb{Z}$

FIRST REMARK: DECOMPOSE THE OPERATOR

How to compute $\bigotimes_{\ell=1}^{n}\left(\frac{|0\rangle+e^{2i\pi k2^{-\ell}}|1\rangle}{\sqrt{2}}\right)$?

Idea: write the binary decomposition of $k2^{-\ell}$

$$
\begin{aligned}
e^{2i\pi k2^{-\ell}} &= e^{2i\pi\left(\sum\limits_{m=1}^{n} 2^{n-m-\ell}k_m\right)} \\
&= e^{2i\pi\left(\sum\limits_{m=n-\ell+1}^{n} 2^{n-m-\ell}k_m\right)} \qquad (\text{if } m \leq n-\ell, \text{ then } 2^{n-m-\ell} \in \mathbb{N}) \\
&= e^{2i\pi\left(\sum\limits_{m=1}^{\ell} 2^{-m}k_{n-\ell+m}\right)} \qquad (n-\ell-m_{\text{old}} \longleftrightarrow -m_{\text{new}})
\end{aligned}
$$

$$
\frac{|0\rangle + e^{2i\pi k2^{-\ell}}|1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{2i\pi 0.k_{n-\ell+1}\cdots k_n}|1\rangle}{\sqrt{2}}
$$

where for any integer $j = j_1 \cdots j_p$

$$
0.j_1\cdots j_p \overset{\text{def}}{=} \frac{j}{2^p} = \sum_{\ell=1}^{p} 2^{-\ell} j_\ell
$$

TO SUMMARIZE

$\mathrm{QFT}_{\mathbb{Z}/2^n\mathbb{Z}} |k\rangle$ is equal to:

$$\left(\frac{|0\rangle + e^{2i\pi 0.k_n} |1\rangle}{\sqrt{2}}\right) \bigotimes \left(\frac{|0\rangle + e^{2i\pi 0.k_{n-1}k_n} |1\rangle}{\sqrt{2}}\right) \bigotimes \cdots \bigotimes \left(\frac{|0\rangle + e^{2i\pi 0.k_1 k_{n-\ell}\cdots k_n} |1\rangle}{\sqrt{2}}\right)$$

where

$$k = \sum_{\ell=1}^{n} 2^{n-\ell} k_\ell \in [\![0, 2^n - 1]\!] \quad \text{and} \quad 0.k_n \cdots k_{n+1-p} = \sum_{\ell=1}^{p} 2^{-\ell} k_{n+1-\ell} \in [0, 1)$$

To build this quantum state, we will crucially used that

$$\text{C-R}_s |b\rangle |1\rangle = |b\rangle e^{\frac{2i\pi b}{2^s}} |1\rangle = |b\rangle e^{2i\pi 0.0^{s-1}b} |1\rangle \quad \text{where} \quad 0.0^{s-1}b = 0.\underbrace{0\ldots 0b}_{s \text{ times}}$$

$$\text{C-R}_s |b\rangle |0\rangle = |b\rangle |0\rangle$$

THE CASE 2^3

Aim: starting from $|k_1, k_2, k_3\rangle$ building

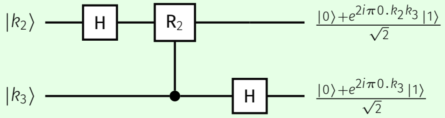
$$\left(\frac{|0\rangle + e^{2i\pi 0.k_3}\,|1\rangle}{\sqrt{2}}\right) \bigotimes \left(\frac{|0\rangle + e^{2i\pi 0.k_2 k_3}\,|1\rangle}{\sqrt{2}}\right) \bigotimes \left(\frac{|0\rangle + e^{2i\pi 0.k_1 k_2 k_3}\,|1\rangle}{\sqrt{2}}\right)$$

1. Sending $|k_3\rangle$ through **H**:

$$|k_3\rangle \xrightarrow{\mathsf{H}} \frac{|0\rangle + (-1)^{k_3}\,|1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{2i\pi 0.k_3}\,|1\rangle}{\sqrt{2}} \qquad \left(0.k_3 = 0 \text{ if } k_3 = 0 \text{ or } \tfrac{1}{2} \text{ if } k_3 = 1\right)$$

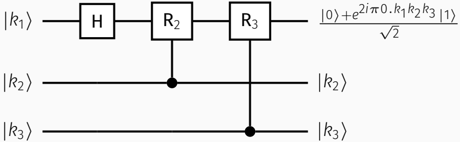
2. Sending $|k_3\rangle\,|k_2\rangle$ through $\mathbf{I}_2 \otimes \mathbf{H}$ and then **C-R**$_2$:

$$|k_3\rangle\,|k_2\rangle \xrightarrow{\mathsf{I}_2 \otimes \mathsf{H}} |k_3\rangle\,\frac{|0\rangle + e^{2i\pi 0.k_2}\,|1\rangle}{\sqrt{2}} \xrightarrow{\mathsf{C}\text{-}\mathsf{R}_2} |k_3\rangle\,\frac{|0\rangle + e^{2i\pi 0.0k_3}\,e^{2i\pi 0.k_2}\,|1\rangle}{\sqrt{2}} = |k_3\rangle\,\frac{|0\rangle + e^{2i\pi 0.k_2 k_3}\,|1\rangle}{\sqrt{2}}$$



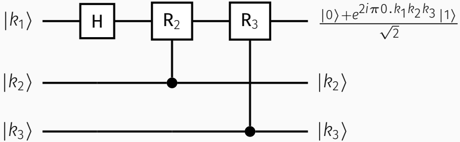
THE CASE 2^3

3. Sending $|k_3\rangle\,|k_2\rangle\,|k_1\rangle$ through the following circuit:



Combining this with the previous circuit gives almost the good state not in the good order: swap!

3. Sending $|k_3\rangle\,|k_2\rangle\,|k_1\rangle$ through the following circuit:



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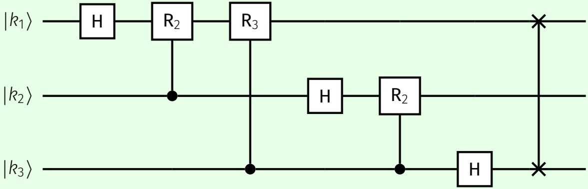
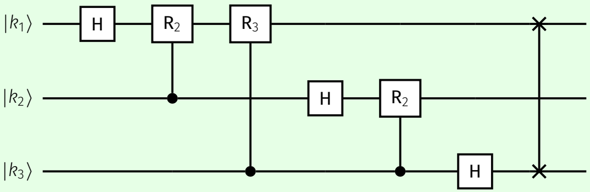
QFT over $\mathbb{Z}/8\mathbb{Z}$



QFT over $\mathbb{Z}/8\mathbb{Z}$



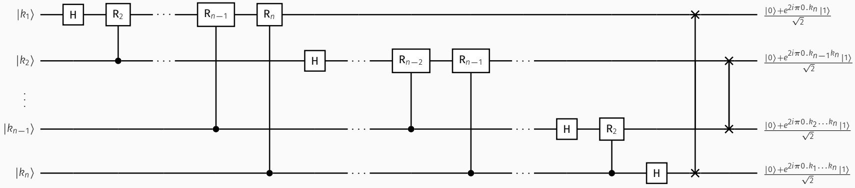
The general case $\mathbb{Z}/2^n\mathbb{Z}$ will follow the same pattern: $O(n^2) + \textbf{SWAP} = O(n^2) = O\left(\log 2^n\right)^2$ gates

$\longrightarrow$ In particular gates $R_2, \cdots, R_n$ are used!

But $\mathbf{R}_s = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2i\pi}{2^s}} \end{pmatrix}$ is very close to the identity if $s \gg \log n$.

If one allows errors: removing all the $\mathbf{R}_s$ for $s \geq C \log n$ (with C constant) will lead to the result with accuracy $\leq \frac{1}{n}$

$\longrightarrow$ In that case: only $O(n \log n)$ gates!



$|k_1\rangle$
$|k_2\rangle$
$|k_{n-1}\rangle$
$|k_n\rangle$
H
R_2
R_{n-1}
R_n
R_{n-2}
$\frac{|0\rangle + e^{2i\pi 0.k_n}|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle + e^{2i\pi 0.k_{n-1}k_n}|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle + e^{2i\pi 0.k_2\dots k_n}|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle + e^{2i\pi 0.k_1\dots k_n}|1\rangle}{\sqrt{2}}$

EXERCISE SESSION