LECTURE 1
INTRODUCTION TO QUANTUM COMPUTING

INF587 Quantum computer science and applications

Thomas Debris-Alazard

Inria, École Polytechnique

ORIGIN OF QUANTUM COMPUTING

Feynman (1981):

Can quantum systems be probabilistically simulated by a classical computer?

⟶ The answer is almost certainly, no!

⟶ Use quantum systems/computers to simulate quantum systems!

$\Big($birth of quantum simulation$\Big)$

A natural question:

What other problems can quantum computers solve more quickly than classical computer?

Deutsch (1985):

Foundation of quantum computing!

⟶ Deutsch-Jozsa algorithm (1992) quantum algorithm faster than any classical algorithm

EARLY ALGORITHMS: SHOR

Shor (1994):

Solves the discrete logarithm and factoring problem efficiently with a quantum computer!

Terrible situation: public-key cryptography currently deployed is broken by using an “efficient” quantum computer

⟶ Cryptographic community worried about this since many years. . .

There exists quantum resistant solutions: post-quantum cryptography (active research topic)

American’s government (2017 & 2023) has launched processes to standardized post-quantum cryptosystems

EARLY ALGORITHMS: GROVER

Grover (1996):

Find an element in a list of size n in time $O\left(\sqrt{n}\right)$ while any classical algorithm needs a time $\approx n$

Consequence: size of keys in symmetric cryptography has to be $\times 2$.

$\Big($ size of cryptosystem ℓ bits: best classical attack costs $2^{\ell} \overset{\text{(Grover)}}{\longmapsto} 2^{\ell/2}\Big)$

A BIG ISSUE: DECOHERENCE

Computations are “noisy”

- Quantum bits are very fragile, they quickly interfere with the environment: decoherence
- Quantum architectures are not “ideal”

⟶ Faults in computation can theoretically be “corrected”: quantum error correcting codes

Theorem [Aharonov, Ben-Or, 1997]:

Quantum computation is possible provided the noise is sufficiently low

QUANTUM CRYPTOGRAPHY

Benett-Brassard (1984):

Quantum protocol for key-exchange

- Already implemented
- If an authenticated canal has been established, unconditional security: relies strongly on the validity of physic laws and not computational assumptions

PROGRAM OF THIS COURSE

⟶ Basics of quantum computing and quantum information theory

- Quantum formalism with density operators, general measures, partial trace, etc. . .
- Quantum circuit model, quantum algorithms (Deutsch-Josza, Simon, Grover, Quantum Fourier Transform, Shor, Kitaev)
- Basics of quantum error correcting codes and quantum cryptography

References:

- Nielsen and Chuang, *Quantum computation and quantum information*,

  ⟶ Nice introduction to quantum computing and quantum information
- de Wolf's lecture notes: `https://arxiv.org/abs/1907.09415`,

  ⟶ Nice for advanced quantum algorithms
- Childs's lecture notes: `https://www.cs.umd.edu/~amchilds/qa/`,

  ⟶ Nice for advanced topics
- Zemor's lecture notes: `https://www.math.u-bordeaux.fr/~gzemor/QuantumCodes.pdf`,

  ⟶ Introduction to quantum error correcting codes

EVALUATION OF THIS COURSE

1. An exam (3 hours): **an $A3$ sheet allowed**

   ⟶ Three exercises seen during the Exercise Sessions will be at the exam

2. Presentation of a research article or a chapter of some lecture notes (30min)

A WARNING

You are in a course of computer science

Computer science: art of computing

⟶ We don't care that an object "exists", we want to compute it efficiently!

Using the law of quantum physic: new model of computation

What does mean quantum computing? What is a quantum algorithm?

⟶ This course is not about the law of physics or about the "technologies" to verify/use them

CLASSICAL BITS VERSUS QUANTUM BITS

CLASSICAL BIT

- Classical bit: $b \in \{0,1\}$ with XOR operation ($1 \oplus 1 = 0 \oplus 0 = 0$ and $1 \oplus 0 = 0 \oplus 1 = 1$)
- Probabilistic bit: $\begin{pmatrix} p \\ q \end{pmatrix}$ where

$$p \overset{\text{def}}{=} \mathbb{P}(b=0)$$

$$q \overset{\text{def}}{=} \mathbb{P}(b=1)$$

- Evolution during a computation (a probabilistic bit stays a probabilistic bit):

$$\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{where} \begin{cases} a+c=1 \\ b+d=1 \end{cases} \quad \text{and } a,b,c,d \geq 0.$$

Probabilistic computation: multiplication by a stochastic matrix

Examples: $b \to b \oplus b$ **and** $b \mapsto b \oplus 1$

$$\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

QUANTUM BIT (QUBIT)

"A superposition of classical states"

- A qubit $|\psi\rangle$ is an element of $\mathbb{C}^2$ with Euclidean norm 1:

  $$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \text{ with } \alpha, \beta \in \mathbb{C} \text{ (called amplitude) and } |\alpha|^2 + |\beta|^2 = 1$$

  where $(|0\rangle, |1\rangle)$ orthonormal basis of $\mathbb{C}^2$. Usually defined as

  $$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

QUANTUM BIT (QUBIT)

"A superposition of classical states"

- A qubit $|\psi\rangle$ is an element of $\mathbb{C}^2$ with Euclidean norm 1:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \text{ with } \alpha, \beta \in \mathbb{C} \text{ (called amplitude) and } |\alpha|^2 + |\beta|^2 = 1$$

where $(|0\rangle\,, |1\rangle)$ orthonormal basis of $\mathbb{C}^2$. Usually defined as

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

We "cannot see" a superposition, we "can only see" classical states: measure and observe!

- Measurement: probabilistic orthogonal projection. Given $|e_0\rangle\,, |e_1\rangle \in \mathbb{C}^2$ orthonormal basis:

$$\text{Measuring in the basis } (|e_0\rangle\,, |e_1\rangle): \ |\psi\rangle = \alpha\,|e_0\rangle + \beta\,|e_1\rangle \xrightarrow{measure} \begin{cases} |e_0\rangle \text{ with prob. } |\alpha|^2 \\ |e_1\rangle \text{ with prob. } |\beta|^2 \end{cases}$$

Exercise: Computational versus Hadamard basis

1. Show that $(|+\rangle\,, |-\rangle)$ is an orthonormal basis of $\mathbb{C}^2$ where

$$|+\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad \text{and} \quad |-\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

2. Give the outcome distribution when measuring $|0\rangle$, $|-\rangle$, and $\frac{1}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle$ in the bases $(|0\rangle\,, |1\rangle)$ and $(|+\rangle\,, |-\rangle)$.

QUANTUM BIT (QUBIT)

- **Qubit**: $|\psi\rangle \in \mathbb{C}^2$ of Hermitian norm 1,
- **Measuring** in the orthonormal basis $(|e_0\rangle, |e_1\rangle)$:

$$|\psi\rangle = \alpha |e_0\rangle + \beta |e_1\rangle \xrightarrow{\text{measure}} \begin{cases} |e_0\rangle \text{ with prob. } |\alpha|^2 \\ |e_1\rangle \text{ with prob. } |\beta|^2 \end{cases}$$

A measurement is a **computation** you have access to

$\longrightarrow$ See Lecture 2 for a precise definition of measurement. . .

QUANTUM BIT (QUBIT)

- **Qubit**: $|\psi\rangle \in \mathbb{C}^2$ **of Hermitian norm 1**,
- **Measuring** in the orthonormal basis $(|e_0\rangle, |e_1\rangle)$:

$$|\psi\rangle = \alpha |e_0\rangle + \beta |e_1\rangle \xrightarrow{\textit{measure}} \begin{cases} |e_0\rangle \text{ with prob. } |\alpha|^2 \\ |e_1\rangle \text{ with prob. } |\beta|^2 \end{cases}$$

A measurement is a **computation** you have access to

⟶ See Lecture 2 for a precise definition of measurement. . .

Are there other computations over qubits we have access to?

QUANTUM BIT (QUBIT)

- **Qubit**: $|\psi\rangle \in \mathbb{C}^2$ of Hermitian norm 1,
- **Measuring** in the orthonormal basis $(|e_0\rangle, |e_1\rangle)$:

$$|\psi\rangle = \alpha |e_0\rangle + \beta |e_1\rangle \xrightarrow{\textit{measure}} \begin{cases} |e_0\rangle \text{ with prob. } |\alpha|^2 \\ |e_1\rangle \text{ with prob. } |\beta|^2 \end{cases}$$

A measurement is a computation you have access to

⟶ See Lecture 2 for a precise definition of measurement. . .

Are there other computations over qubits we have access to?

⟶ Yes! Unitary evolutions

UNITARY EVOLUTIONS

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}, \text{ then its conjugate transpose } \mathbf{U}^\dagger = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

- Unitary evolution: $\mathbf{U} \in \mathbb{C}^{2\times 2}$ unitary matrix $\iff \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}_2$

$$|\psi\rangle \longrightarrow \mathbf{U}\,|\psi\rangle$$

Is it true that a qubit is still a qubit after a unitary evolution? Why?

UNITARY EVOLUTIONS

$$\mathsf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}, \text{ then its conjugate transpose } \mathsf{U}^\dagger = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

- Unitary evolution: $\mathsf{U} \in \mathbb{C}^{2\times 2}$ **unitary matrix** $\iff$ $\mathsf{U}\mathsf{U}^\dagger = \mathsf{I}_2$

$$|\psi\rangle \longrightarrow \mathsf{U}\,|\psi\rangle$$

Is it true that a qubit is still a qubit after a unitary evolution? Why?

$\longrightarrow$ Yes! Unitary evolutions preserve the Hermitian norm (more generally the Hermitian product)

Unitary evolutions are invertible!

$$|\psi\rangle \xrightarrow{\mathsf{U}} \mathsf{U}\,|\psi\rangle \xrightarrow{\mathsf{U}^\dagger} \mathsf{U}^\dagger\mathsf{U}\,|\psi\rangle = |\psi\rangle$$

- $\mathsf{U} \in \mathbb{C}^{2\times 2}$ unitary over qubits is often called **quantum gate**

  $\longrightarrow$ It exists a small set of gates which is **universal** (be patient, wait Lecture 4)

QUANTUM GATES

To define a quantum gate: enough to specify the image of an orthonormal basis and then extended it by linearity

But it has to map an orthonormal basis to an orthonormal basis!

Exercise: Quantum Gates?

Are the following linear operators over qubits be quantum gates?

1. $|0\rangle \mapsto |1\rangle$ and $|1\rangle \mapsto \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$,
2. $|0\rangle \mapsto |1\rangle$ and $|1\rangle \mapsto |0\rangle$.

Quantum gates have matrix representations!

For instance: $|0\rangle \mapsto |1\rangle$ and $|1\rangle \mapsto |0\rangle$ has the representation: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Only linear operator that maps $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle$ to $|0\rangle$.

EXAMPLE OF QUANTUM GATES

- NOT-gate **X**:

Linear op.	Matrix rep.
$\lvert 0\rangle \mapsto \lvert 1\rangle$ $\lvert 1\rangle \mapsto \lvert 0\rangle$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- Hadamard-gate **H**:

Linear op.	Matrix rep.
$\lvert 0\rangle \mapsto \frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$ $\lvert 1\rangle \mapsto \frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Exercise:

1. What is the effect of applying **H** on $\lvert 0\rangle$ and measuring it?
2. What is the effect of applying **H** on $\lvert 0\rangle$ twice?

CLASSICAL VERSUS QUANTUM COMPUTATION

Is quantum computation over qubits the same than classical computation over probabilistic bits?

Exercise:

Show that there is no stochastic matrix $\mathbf{P}$ which when applied to 0, *i.e.* to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, simulates the effect of the Hadamard gate

The "-1" gives you a huge power. . .

YOUR FIRST QUANTUM ALGORITHM

THE DEUTSCH-JOSZA PROBLEM

Problem:

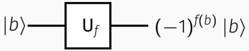
- Input: $f : \{0,1\}^n \to \{0,1\}$ either constant or balanced
- Output: 0 if and only if f is constant

Query complexity to f to find the correct answer with certainty:

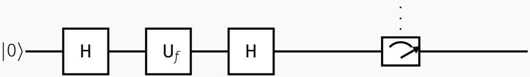
- Classically: $1 + \frac{2^n}{2}$
- Quantumly: 1

THE DEUTSCH-JOSZA ALGORITHM FOR $n = 1$

- Suppose that we have access to the following gate (see exercise session)



- The algorithm



in the basis $(|0\rangle\,, |1\rangle)$

- Analysis

1. Applying **H**: $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

2. Applying $\mathbf{U}_f$:

$$\mathsf{U}_f\left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right) = \frac{1}{\sqrt{2}}\left(\mathsf{U}_f\,|0\rangle + \mathsf{U}_f\,|1\rangle\right) = \frac{(-1)^{f(0)}\,|0\rangle + (-1)^{f(1)}\,|1\rangle}{\sqrt{2}}$$

3. Applying **H**:

$$\mathsf{H}\left(\frac{(-1)^{f(0)}\,|0\rangle + (-1)^{f(1)}\,|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left((-1)^{f(0)}\mathsf{H}\,|0\rangle + (-1)^{f(1)}\mathsf{H}\,|1\rangle\right)$$

$$= \frac{\left((-1)^{f(0)} + (-1)^{f(1)}\right)|0\rangle + \left((-1)^{f(0)} - (-1)^{f(1)}\right)|1\rangle}{2}$$

CORRECTION OF THE DEUTSCH-JOSZA ALGORITHM FOR $n = 1$

Before measuring we have computed:

$$|\psi_{\text{out}}\rangle \overset{\text{def}}{=} \frac{\left((-1)^{f(0)} + (-1)^{f(1)}\right)|0\rangle + \left((-1)^{f(0)} - (-1)^{f(1)}\right)|1\rangle}{2}$$

- If f constant:

$$|\psi_{\text{out}}\rangle = \pm |0\rangle$$

- If f balanced, namely $f(0) \neq f(1)$:

$$|\psi_{\text{out}}\rangle = \pm |1\rangle$$

CORRECTION OF THE DEUTSCH-JOSZA ALGORITHM FOR $n = 1$

Before measuring we have computed:

$$|\psi_{\text{out}}\rangle \overset{\text{def}}{=} \frac{\left((-1)^{f(0)} + (-1)^{f(1)}\right)|0\rangle + \left((-1)^{f(0)} - (-1)^{f(1)}\right)|1\rangle}{2}$$

- If f constant:

$$|\psi_{\text{out}}\rangle = \pm|0\rangle$$

- If f balanced, namely $f(0) \neq f(1)$:

$$|\psi_{\text{out}}\rangle = \pm|1\rangle$$

Measuring in the $(|0\rangle, |1\rangle)$ basis leads to (with probability one)

$|0\rangle$ if f constant or $|1\rangle$ if f balanced

n QUBITS SYSTEM

FINITE DIMENSION

During all this course we will work in finite dimension, think $\mathbb{C}^N$

$\longrightarrow$ Vector spaces have finite dimension, linear operators can be written as matrices, etc. . .

TENSOR PRODUCT

Given two vector spaces V and W, the tensor product $\mathbf{v} \otimes \mathbf{w}$ between $\mathbf{v} \in V$ and $\mathbf{w} \in W$ verifies:

(1) for any scalar z,

$$z\,(\mathbf{v} \otimes \mathbf{w}) = (z\mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (z\mathbf{w})$$

(2) for any $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$(\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} = \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w}$$

(3) for any $\mathbf{w}_1, \mathbf{w}_2 \in W$,

$$\mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \otimes \mathbf{w}_1 + \mathbf{v} \otimes \mathbf{w}_2$$

The tensor product $\mathbf{v} \otimes \mathbf{w}$ as a column/row product:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \begin{pmatrix} w_1 & \cdots & w_{N'} \end{pmatrix}$$

TENSOR PRODUCT OF SPACES

Tensor product of spaces:

V and W be two vector spaces with bases the $\mathbf{v}_i$'s and the $\mathbf{w}_j$ respectively

$$V = \operatorname{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n) \quad \text{and} \quad W = \operatorname{Span}(\mathbf{w}_1, \dots, \mathbf{w}_m)$$

The vector space $V \otimes W$ is defined as being generated by the $\mathbf{v}_i$'s and the $\mathbf{w}_j$'s

$$V \otimes W \overset{\text{def}}{=} \operatorname{Span}\left(\mathbf{v}_i \otimes \mathbf{w}_j \ : \ 1 \leq i \leq n, \ 1 \leq j \leq m\right)$$

- **Dimension**, (multiplicative)

$$\mathbf{dim}\, V \otimes W = \mathbf{dim}\, V\, \mathbf{dim}\, W = nm$$

- **Basis**, $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ (*resp.* $(\mathbf{w}_1, \dots, \mathbf{w}_m)$) be a basis of V (*resp.* W)

$$\left(\mathbf{v}_i \otimes \mathbf{w}_j : 1 \leq i \leq n, \ 1 \leq j \leq m\right) \ \text{is a basis of } V \otimes W$$

- **Characterization**

$$\mathbf{x} \in V \otimes W \iff \exists \alpha_{i,j} \ : \ \mathbf{x} = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \alpha_{i,j}\, \mathbf{v}_i \otimes \mathbf{w}_j$$

Classical error:

$\mathbf{x} \in V \otimes W$, then there exists $\mathbf{v} \in V$ and $\mathbf{w} \in W$ such that $\mathbf{x} = \mathbf{v} \otimes \mathbf{w}$.

SCALAR PRODUCT OVER TENSOR PRODUCT SPACES

$(\mathbf{v}_1, \ldots, \mathbf{v}_n) \Big(\textit{resp.}\ (\mathbf{w}_1, \ldots, \mathbf{w}_m) \Big)$ be a basis of V (*resp.* W).

Scalar product over tensor product spaces:

Suppose that V (*resp.* W) is equipped by a scalar product $\langle \cdot, \cdot \rangle_V$ (*resp.* $\langle \cdot, \cdot \rangle_W$). The scalar product over $V \otimes W$ is defined as (and extended by bilinearity) as

$$\langle \mathbf{v}_i \otimes \mathbf{w}_j, \mathbf{v}_k \otimes \mathbf{w}_\ell \rangle_{V \otimes W} \overset{\text{def}}{=} \langle \mathbf{v}_i, \mathbf{v}_k \rangle_V \ \langle \mathbf{w}_j, \mathbf{w}_\ell \rangle_W$$

An important remark:

If $\mathbf{v}_1 \perp \mathbf{v}_2$, then for all $\mathbf{w}_1, \mathbf{w}_2$: $\quad (\mathbf{v}_1 \otimes \mathbf{w}_1) \perp (\mathbf{v}_2 \otimes \mathbf{w}_2)$

LINEAR OPERATOR OVER TENSOR PRODUCT OF SPACES

$$(\mathbf{v}_1, \ldots, \mathbf{v}_n) \left(\textit{resp.}\ (\mathbf{w}_1, \ldots, \mathbf{w}_m) \right) \text{ be a basis of } V\ (\textit{resp. } W).$$

Linear operator over tensor product of spaces:

Given $\mathbf{A}$, $\mathbf{B}$ be linear operators over V, W, $\mathbf{A} \otimes \mathbf{B}$ is a linear operator over $V \otimes W$ be defined (and extended by linearity) as

$$\mathbf{A} \otimes \mathbf{B} \left(\mathbf{v}_i \otimes \mathbf{w}_j \right) \overset{\text{def}}{=} \mathbf{A}\mathbf{v}_i \otimes \mathbf{B}\mathbf{w}_j$$

- Characterization,

$$\mathbf{C} \text{ linear operator over } V \otimes W \iff \exists \alpha_i, \mathbf{A}_i, \mathbf{B}_i \ : \ \mathbf{C} = \sum_i \alpha_i \ \mathbf{A}_i \otimes \mathbf{B}_i$$

Classical error:

$\mathbf{C}$ linear operator over $V \otimes W$, then there exists $\mathbf{A}$, $\mathbf{B}$ linear operators over V and W s.t $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$.

MATRIX REPRESENTATION OF TENSOR PRODUCT

Tensor product of matrices:

Let $\mathsf{A} \overset{\text{def}}{=} (a_{i,j})_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \in \mathbb{C}^{n\times m}$ and $\mathsf{B} \in \mathbb{C}^{p\times q}$, then

$$\mathsf{A} \otimes \mathsf{B} \overset{\text{def}}{=} \begin{pmatrix} a_{1,1}\mathsf{B} & a_{1,2}\mathsf{B} & \cdots & a_{1,m}\mathsf{B} \\ a_{2,1}\mathsf{B} & a_{2,2}\mathsf{B} & \cdots & a_{2,m}\mathsf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}\mathsf{B} & a_{n,2}\mathsf{B} & \cdots & a_{n,m}\mathsf{B} \end{pmatrix} \in \mathbb{C}^{np\times mq}$$

Example:

1. $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\times 2 \\ 1\times 3 \\ 2\times 2 \\ 2\times 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}.$
2. $\mathsf{X} \otimes \mathsf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

PROPERTIES OF THE TENSOR PRODUCT OF MATRICES

Properties:

For any $\alpha \in \mathbb{C}$, $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m \times n}$ and $\mathbf{C}, \mathbf{D} \in \mathbb{C}^{p \times q}$

1. $\alpha\left(\mathbf{A} \otimes \mathbf{C}\right) = \left(\alpha\mathbf{A}\right) \otimes \mathbf{C} = \mathbf{A} \otimes \left(\alpha\mathbf{C}\right)$
2. $\left(\mathbf{A} + \mathbf{B}\right) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$
3. $\mathbf{C} \otimes \left(\mathbf{A} + \mathbf{B}\right) = \mathbf{C} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{B}$
4. If we can form matrices products **AC** and **BD**, then
$$\left(\mathbf{A} \otimes \mathbf{B}\right)\ \left(\mathbf{C} \otimes \mathbf{D}\right) = \left(\mathbf{AC}\right) \otimes \left(\mathbf{BD}\right)$$
5. If **A**, **B** are invertible, then $\qquad \left(\mathbf{A} \otimes \mathbf{B}\right)^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}.$

Classical error:

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{B} \otimes \mathbf{A}$$

n QUBITS SYSTEM

- A qubit $|\psi\rangle$ is an element of $\mathbb{C}^2$ with Hermitian norm 1,
- A **register of n qubits** $|\psi\rangle$ is an element of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$ with Euclidean norm 1.

Let $(|0\rangle, |1\rangle)$ be an orthonormal basis of $\mathbb{C}^2$. Then,

$$(|b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle \ : \ b_1, \ldots, b_n \in \{0,1\})$$

is an orthonormal basis of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$

- Notation: for $b_1, \ldots, b_n \in \{0,1\}$ and $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ be qubits

  $|b_1 b_2 \ldots b_n\rangle \overset{\text{def}}{=} |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle$ and $|\psi_1\rangle |\psi_2\rangle \ldots |\psi_n\rangle \overset{\text{def}}{=} |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$

- Characterization: any register $|\psi\rangle \in \mathbb{C}^{2^n}$ of n qubits can be written as

$$|\psi\rangle = \sum_{\mathbf{x} \in \{0,1\}^n} \alpha_{\mathbf{x}} |\mathbf{x}\rangle \quad \text{where } \alpha_{\mathbf{x}} \in \mathbb{C} \text{ (called amplitude)} \quad \text{and} \quad \sum_{\mathbf{x} \in \{0,1\}^n} |\alpha_{\mathbf{x}}|^2 = 1$$

n QUBITS SYSTEM

- A qubit $|\psi\rangle$ is an element of $\mathbb{C}^2$ with Hermitian norm 1,
- A register of n qubits $|\psi\rangle$ is an element of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$ with Euclidean norm 1.

Let $(|0\rangle, |1\rangle)$ be an orthonormal basis of $\mathbb{C}^2$. Then,

$$(|b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle \ : \ b_1, \ldots, b_n \in \{0,1\})$$

is an orthonormal basis of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$

- Notation: for $b_1, \ldots, b_n \in \{0,1\}$ and $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ be qubits

  $|b_1 b_2 \ldots b_n\rangle \overset{\text{def}}{=} |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle$ and $|\psi_1\rangle |\psi_2\rangle \ldots |\psi_n\rangle \overset{\text{def}}{=} |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$

- Characterization: any register $|\psi\rangle \in \mathbb{C}^{2^n}$ of n qubits can be written as

  $$|\psi\rangle = \sum_{\mathbf{x} \in \{0,1\}^n} \alpha_{\mathbf{x}} |\mathbf{x}\rangle \quad \text{where } \alpha_{\mathbf{x}} \in \mathbb{C} \text{ (called amplitude)} \quad \text{and} \quad \sum_{\mathbf{x} \in \{0,1\}^n} |\alpha_{\mathbf{x}}|^2 = 1$$

A remark: choose your orthonormal basis!

From any $(|e_0\rangle, |e_1\rangle)$ orthonormal basis of $\mathbb{C}^2$, then $\left(\left|e_{i_1}\right\rangle \ldots \left|e_{i_n}\right\rangle\right)$ for $i_1, \ldots, i_n \in \{0,1\}^n$ is an orthonormal basis of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$

Exercise:

1. Compute the scalar product between $|+\rangle\,|1\rangle$, $|00\rangle$ and $|11\rangle$ where $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$.
2. Let $(|e_0\rangle\,,|e_1\rangle)$ be an orthonormal basis of $\mathbb{C}^2$. Show that $\left(\left|e_{i_1}\right\rangle \ldots \left|e_{i_n}\right\rangle\right)$ for $i_1,\ldots,i_n \in \{0,1\}^n$ is an orthonormal basis of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$.
3. Do we have $|00\rangle + |10\rangle = (|0\rangle + |1\rangle) \otimes |0\rangle$?
4. (*) Do there exist two qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ such that
$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\psi_1\rangle \otimes |\psi_2\rangle\,.$$
5. Do there exist two qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ such that
$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = |\psi_1\rangle \otimes |\psi_2\rangle\,.$$

SEPARABLE VERSUS ENTANGLED STATES

Separable versus entangled states:

A n-qubit system $|\psi\rangle$ that can be decomposed as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ is called separable.

When there is no such decomposition, the state is called entangled.

Example:

1. Separable states

$$|00\rangle = |0\rangle \otimes |0\rangle \quad \text{and} \quad \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

2. Entangled state

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$\longrightarrow$ Entangled states play a crucial role in quantum computation/information (teleportation, quantum cryptography, . . .)

MEASURING n QUBITS SYSTEM

- Measuring in the basis $|e_1\rangle\,|e_2\rangle\cdots|e_n\rangle$:

$$|\psi\rangle = \sum_{i_1,\dots,i_n\in\{0,1\}^n} \alpha_{i_1\dots i_n}\left|e_{i_1}\right\rangle\cdots\left|e_{i_n}\right\rangle \xrightarrow{\textit{measure}} \left|e_{j_1}\right\rangle\cdots\left|e_{j_n}\right\rangle \text{ with probability } |\alpha_{j_1\dots j_n}|^2$$

- Measuring the first register in the basis $(|e_0\rangle\,,|e_1\rangle)$

$$|\psi\rangle = \alpha_0\,|e_0\rangle\,|\psi_0\rangle + \alpha_1\,|e_1\rangle\,|\psi_1\rangle \xrightarrow{\textit{measure}} \begin{cases} |e_0\rangle\,|\psi_0\rangle \text{ with prob. } |\alpha_0|^2 \\ |e_1\rangle\,|\psi_1\rangle \text{ with prob. } |\alpha_1|^2 \end{cases}$$

Be careful: necessarily $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

Exercise:

Give the outcome distribution of measuring in the basis $(|bb'\rangle : b, b' \in \{0,1\})$ the first registers of the following two-qubits

$$|0\rangle\left(\sqrt{\frac{1}{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle\right), \quad \sqrt{\frac{1}{2}}\,|01\rangle + \sqrt{\frac{1}{3}}\,|11\rangle + \sqrt{\frac{1}{6}}\,|10\rangle \quad \text{and} \quad \frac{1}{2}\left(|0\rangle - |1\rangle\right)\left(|0\rangle - |1\rangle\right)$$

UNITARY FOR n QUBITS SYSTEMS

Unitary evolution $\mathbf{U} \in \mathbb{C}^{2^n \times 2^n}$ unitary matrix $\iff \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}_{2^n}$

Exercise:

Is the following operator a unitary of $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Describe the image of $\left|bb'\right\rangle$ for $b, b' \in \{0, 1\}$

BRA-KET AND KET-BRA NOTATION

THE BRA-KET NOTATION

Scalar Product:

Let $|e_1\rangle, \ldots, |e_{2^n}\rangle$ be an orthonormal basis, $|\psi\rangle \overset{\text{def}}{=} \sum_i \alpha_i |e_i\rangle$ and $|\varphi\rangle \overset{\text{def}}{=} \sum_i \beta_i |e_i\rangle$. Then

$$\langle\psi|\varphi\rangle \overset{\text{def}}{=} \sum_i \overline{\alpha_i}\beta_i.$$

- **Ket-notation:** $|\psi\rangle$ is called a ket
- **Bra-notation:** a ket $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2^n} \end{pmatrix}$ is a vector of $\mathbb{C}^{2^n}$,

$\langle\psi| \overset{\text{def}}{=} (|\psi\rangle)^\dagger = \begin{pmatrix} \overline{\alpha_1} & \ldots & \overline{\alpha_{2^n}} \end{pmatrix}$ is a bra (don't forget the conjugate, $\overline{\alpha_i}$, not α_i)

Useful notation:

$\longrightarrow$ It enables to interpret $\langle\psi|\varphi\rangle$ as $\langle\psi| \cdot |\varphi\rangle$

Bra	Ket
$\langle\psi\|$	$\|\psi\rangle$

THE KET-BRA NOTATION:

The $|\varphi\rangle\langle\psi|$ operator:

$$\begin{aligned} |\varphi\rangle\langle\psi| : \left(\mathbb{C}^2\right)^{\otimes n} &\longrightarrow \left(\mathbb{C}^2\right)^{\otimes n} \\ \left|\psi'\right\rangle &\longmapsto |\varphi\rangle\langle\psi| \left|\psi'\right\rangle \overset{\text{def}}{=} \left\langle\psi\middle|\psi'\right\rangle |\varphi\rangle \,. \end{aligned}$$

Exercise:

1. Give the image of $|0\rangle$ and $|1\rangle$ by $|0\rangle\langle 1| + |1\rangle\langle 0|$. Give the matrix representation of this operator. Do you recognize a quantum gate?

2. Let $(|i\rangle)_{i\in\mathcal{I}}$ be an orthonormal basis. Which operator is

$$\sum_{i\in\mathcal{I}} |i\rangle\langle i| ?$$

ADJOINT OF AN OPERATOR:

Adjoint of an operator:

$\mathsf{A}^\dagger$ is known as the adjoint of A

Exercise:

1. Show that $(\mathsf{A}\,|\varphi\rangle)^\dagger = \langle\varphi|\,\mathsf{A}^\dagger$,
2. Show that $(|\varphi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\varphi|$.

AN IMPORTANT PROPERTY

Be careful with adjoint/dagger over tensor product. . . (do not reverse the order. . .)

Proposition:

We have,

$$(|\varphi\rangle\,|\psi\rangle)^\dagger = \langle\varphi|\,\langle\psi| \quad \text{and} \quad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

Proof:

Use the definition of tensor product as multiplication raw/column.

Classical error:

$$(|\varphi\rangle\,|\psi\rangle)^\dagger = \langle\psi|\,\langle\varphi| \quad \text{and} \quad (A \otimes B)^\dagger = B^\dagger \otimes A^\dagger$$

EXERCISE SESSION